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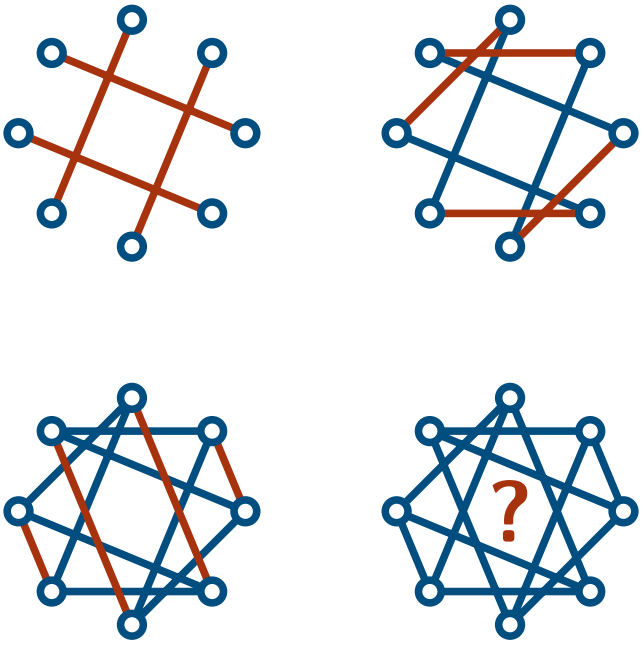
Start Doing Graph Theory

Part IV: Matchings



available online at https://vertex.degree/

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About this document

This is Part IV of *Start Doing Graph Theory*. It introduces bipartite graphs, and then talks about matching theory, first in bipartite graphs and then in general graphs. Many applications of graph theory use matchings and Hall's theorem in particular, so this is an important topic for non-graph theorists to know a little about.

Right now, only parts I through IV are ready. When the book is finished, I plan to provide a variety of ways to read it: a single PDF of the whole book, several smaller PDFs like this one, and eventually an HTML version. For now, think of this document as a preview!

12 Bipartite matching

The purpose of this chapter

In this chapter, we will not yet see any of the big theorems about matchings; before we encounter them in the next chapter, I will explain the choices I made in their presentation. Here, we will only lay the foundations and begin to study bipartite graphs, matchings, and vertex covers.

I have gone to some trouble in this textbook to postpone the definition of bipartite graphs until this chapter. This is done for two reasons. First, I do not want the first few chapters to be overloaded with definitions and nothing but definitions—as much as I can help it, at least. Second, I do not want to give a definition when we have no use for it. This is because, to the extent I can, I want the definitions we make and the questions we ask about them to seem like reasonable definitions we make and reasonable questions to ask.

As a result, it is only now that I define bipartite graphs, because now we can ask the bipartite matching problem. Here, it is reasonable to ask whether (and in which way) a graph is bipartite, and so I hope that the definitions and initial theorems do not feel pointless.

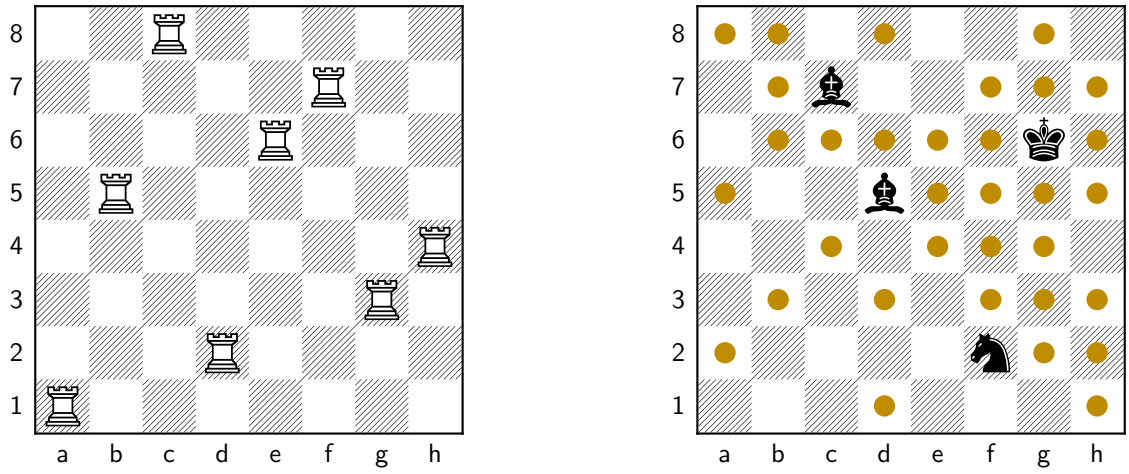
12.1 Rooks on chessboards

In chess, the rook is a piece that can move horizontally or vertically any number of spaces; it is said to "attack" the squares it can move to. How many rooks can be placed on the 8×8 chessboard so that no two of them attack each other? There is a limit of 8, because no two rooks can occupy a rank with the same number. (Similarly, no two rooks can occupy a file with the same letter.) On the other hand, there are many ways to place 8 rooks that do not attack each other: Figure 12.1a shows only one of those ways.

Suppose that we place a few more pieces on the board, as shown in Figure 12.1b. Is it still possible to place 8 rooks, with only one per rank and only one per file,[1] without using either the squares occupied by the existing pieces, or the squares attacked by them? (If you do not know the rules of chess, you can still attempt this puzzle, because I have marked the squares attacked by the black pieces.)

I will not spoil the answer to the puzzle for you; instead, let's discuss how we can model this problem as a graph. There's actually two ways (at least) to do this:

[1] Experienced chess players will observe that in Figure 12.1b, it's sometimes possible to place two rooks in the same rank or file that will not attack each other, because there's an obstacle in the way; for example, on `a7` and `d7`. For the sake of a simpler problem, we will ignore this possibility.



(a) Rooks on an empty chessboard

(b) Placing rooks with obstacles

Figure 12.1: Placing rooks on a chessboard

1. Take the graph whose vertices are the 27 unmarked squares in Figure 12.1b, and place an edge between every pair of squares in the same rank or in the same file.

   Then, the goal is to select 8 vertices in the graph such that no two are adjacent.

2. Take the graph whose vertices are the 8 ranks (1 through 8) as well as the 8 files (a through h), and place an edge between every rank and every file that intersect in an unmarked square.

   Then, the goal is to select 8 edges in the graph such that no two share an endpoint.

It turns out that the first type of problem (which we will study in Chapter 17) is very difficult in general: it is the independent set problem. Meanwhile, the second type of problem, called the bipartite matching problem, is much more tractable to solve. So in this chapter, we will choose the second model.

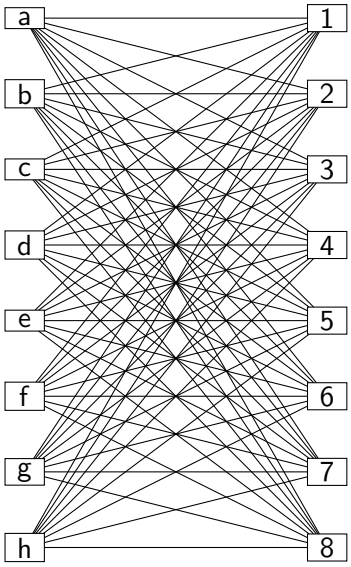
Before we even discuss the bipartite matching problem and how to solve it, though, let's discuss the special structure of the graph in which we have posed it.

12.2 Bipartite graphs

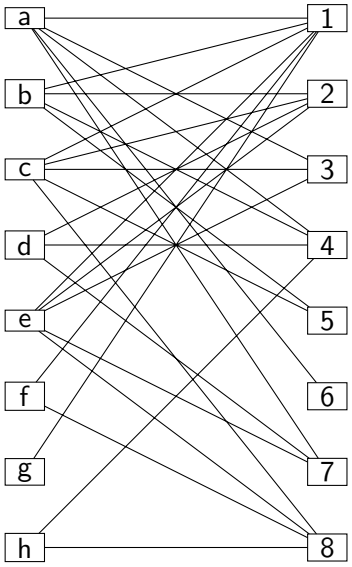
The graph which represents the rook placement problem using vertices for the ranks and the files is a special case of a bipartite graph.

Definition 12.1. *A graph G is called **bipartite** if we can write $V(G) = A \cup B$, where A and B are disjoint sets, such that every edge of G has one endpoint in A and one endpoint in B. The pair (A, B) is called a **bipartition** of G, and the sets A and B are called the two **sides**[2] of the bipartition.*

[2] In graph theory literature, these have no consistent name; they can also be called "parts", or "blocks", or even "partite sets".



(a) Graph for the empty chessboard



(b) Graph for the chessboard with obstacles

Figure 12.2: Placing rooks on a chessboard

In the rook placement problem, we can take the bipartition (A, B) with $A = \{\mathsf{a}, \mathsf{b}, \ldots, \mathsf{h}\}$ and $B = \{1, 2, \ldots, 8\}$. Figure 12.2 shows the two graphs we get for the empty chessboard and for the chessboard with obstacles. As always, we can draw a graph in any way we like; however, when we want to make it visually clear that a graph is bipartite it is traditional to arrange the two sides A and B in two columns (as in Figure 12.2) or in two rows.

The graph in Figure 12.2a representing the empty chessboard is a special case: a complete bipartite graph. (Just as the complete graph has all the edges a graph could possibly have, a complete bipartite graph has all the edges a bipartite graph could possibly have.) Here is one general definition of complete bipartite graphs:

Definition 12.2. *For any $m \geq 1$ and $n \geq 1$, the **complete bipartite graph $K_{m,n}$** is the graph with $m + n$ vertices $\{1, 2, \ldots, m + n\}$ and all mn possible edges xy where $1 \leq x \leq m$ and $m + 1 \leq y \leq m + n$.*

However, we are happy to call any graph isomorphic to $K_{m,n}$ a complete bipartite graph.

In the case of the graphs for the rook placement problem, the bipartition (A, B) is a natural part of the definition of the graph. The two vertices come in two types, representing the ranks and the files of the chessboard, and the rule defining adjacency is a relationship between the two types of vertex. We looked at another bipartite matching problem in Chapter 1, and graph we considered there came with a natural bipartition as well.

We might also encounter bipartite graphs in the wild, and when we do, it might not be obvious that they are bipartite. How do we test if a graph has this property?

In general, a problem that involves labeling the vertices of a graph (such as with two sides A and B) might be difficult because we might have to guess, make mistakes, and backtrack. Here, we are lucky because there are forced deductions: in a bipartite graph, if there is an edge xy

where $x \in A$, then we know that $y \in B$ without guessing. The following algorithm relies only on making such forced deductions.

Given a graph G, we intend to give each vertex a label, A or B, to indicate which side of the bipartition it is on. Initially, all vertices start unlabeled.

1. Pick an arbitrary unlabeled vertex and label it with A.
2. Go through all vertices newly labeled with A, and label all their neighbors with B.
3. Go through all vertices newly labeled with B, and label all their neighbors with A.
4. Repeat steps 2–3 until a vertex has been labeled with both A and B, or until no more unlabeled vertices are being labeled.
5. If the graph is not connected, repeat steps 1–4 for each connected component.

If the graph we are working with is bipartite, then at the end, we get a bipartition (A, B), where A is the set of all vertices labeled with A, and B is the set of all vertices labeled with B.

Question:	What happens if the graph is not bipartite?
Answer:	We will give a vertex both labels, and stop. Such a vertex cannot exist in a bipartite graph, so if we are forced to have such a vertex, then the graph cannot be bipartite.

Question:	Why are we free to label x with A in step 2: at the beginning, and when we consider each new connected component?
Answer:	In each connected component, we can swap the roles of A and B. So for every solution where $x \in B$, there is another solution where $x \in A$.

There is another way of thinking about this algorithm. Compare it to the first algorithm we've studied in this textbook: the distance-finding algorithm in Chapter 3. In that algorithm, we also started at a vertex x, then considered all the neighbors of x, then all of their neighbors, and so on, labeling as we went. The only difference is that when we computed distances, we used the labels $0, 1, 2, 3, \ldots$ at successive stages. In this algorithm, we alternate the label $A, B, A, B, \ldots$ at successive stages, instead.

Question:	Using distances, how can we quickly define what the sets A and B will be in x's connected component?
Answer:	In the end, A will be the set of all vertices at an even distance from x, and B will be the set of all vertices at an odd distance from x.

12.3 Odd cycles

Although the labeling algorithm is a convenient way to test a specific graph for being bipartite, it is not as useful theoretically: when proving a general result, we would like some testable criteria. One way to prove a graph is bipartite is to find a bipartition (A, B) and show that it satisfies the definition, but we'd also like to be able to prove that a graph is not bipartite. For this, we have the following theorem:

Theorem 12.1. *A graph G is bipartite if and only if it contains no **odd cycles**, or cycles of odd length.*

Proof. First, we prove that in a bipartite graph, all cycles must be even.

Let G be bipartite with bipartition (A, B), and let $(x_0, x_1, x_2, \ldots, x_k)$ with $x_0 = x_k$ be a cycle. Without loss of generality, suppose that $x_0 \in A$. Because edges x_0x_1, x_1x_2, and so on must have one endpoint on each side of the bipartition, we must have $x_1 \in B$, $x_2 \in A$, and so on. In general, we can prove by induction that $x_i \in A$ whenever i is even, and $x_i \in B$ whenever i is odd: the base case is x_0, and then induction step simply uses the fact that the endpoints of edge x_ix_{i+1} must be on opposite sides.

However, we also know that $x_k \in A$, because $x_k = x_0$. Therefore k must be even: the cycle has even length. The first half of the proof is complete!

It is easy at this point in the proof to get confused and prove the same thing we just did a second time—for example, this would happen if we continued by proving that if G contains an odd cycle, it is not bipartite. To continue correctly, we will assume that G is not bipartite, and prove that it contains an odd cycle.

The condition "G is not bipartite" is very difficult to work with: it says that a bipartition (A, B) does not exist, or that any candidate pair (A, B) somehow fails to be a bipartition. To make use of this, we define a pair (A, B) that *ought* to be a bipartition in any bipartite graph. Inspired by our labeling algorithm, we begin by choosing a vertex from each connected component of G, and then defining:

- A to be the set of all vertices at an even distance from a chosen vertex;
- B to be the complement of A: the set of all vertices at an odd distance from a chosen vertex.

Since G is not bipartite, this is not a bipartition. So there must be some edge xy between two vertices both in A, or both in B. Since x and y are necessarily in the same connected component, they must both be at an even distance, or both at an odd distance, from the chosen vertex z in that component.

Consider the walk that begins at x, follows some shortest $x - z$ path, then follows some shortest $z - y$ path, then takes the edge xy. The first two segments of this walk both have odd length or both have even length, so combined, their length is even. The final edge xy increases the length by 1. We've found something that's *almost* an odd cycle: it's a closed walk of odd length.

To complete the proof, let $x_0, x_1, \ldots, x_k$ with $x_k = x_0$ be a shortest closed walk of odd length: we will prove that in fact, it is an odd cycle. Suppose not: then there are some positions i and

j with $i < j$ and $x_i = x_j$. (We're not counting the pair $x_0 = x_k$ that exists by definition, so we can assume $j < k$.)

The closed walk $x_i, x_{i+1}, \ldots, x_j$ has length $j - i$; the walk $x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_k$ has length $k - (j - i)$. Since the sum of these two lengths is the odd number k, at least one of these lengths must be odd; since $i < j < k$, both lengths are smaller than k. Therefore we've found an even shorter closed walk of odd length, contradicting our choice of $x_0, x_1, \ldots, x_k$. We conclude that the pair i, j cannot exist: the closed walk we found is an odd cycle! This completes the proof. □

Question:	In the second half of the proof, why can't we skip directly to the odd closed walk $x_0, x_1, \ldots, x_k$?
Answer:	Before we can define a shortest closed walk of odd length, we need to know that the graph contains such closed walks in the first place: in an empty set of walks, there is no shortest walk!

To conclude this section, let's consider two examples of bipartite graphs, and how we prove that they are bipartite. In one case, we will use the definition directly; in the other, we will use Theorem 12.1.

Proposition 12.2. *For all $n \geq 1$, the hypercube graph Q_n is bipartite.*

Proof. The vertices of Q_n are bit strings $b_1 b_2 \ldots b_n$ where each b_i is either 0 or 1. To define a partition (A, B) of $V(Q_n)$, we place a vertex $b_1 b_2 \ldots b_n$ in A if $b_1 + b_2 + \cdots + b_n$ is even, and in B if $b_1 + b_2 + \cdots + b_n$ is odd.

Each edge of Q_n joins two vertices x and y that differ only in one position; $x_i \neq y_i$ for some i, but $x_j = y_j$ for all $j \neq i$. As a result,

$$(x_1 + x_2 + \cdots + x_n) - (y_1 + y_2 + \cdots + y_n) = x_i - y_i = \pm 1 :$$

every difference $x_j - y_j$ with $j \neq i$ cancels. Since the sums differ by ± 1, one of them is even and one is odd: the edge xy has one endpoint in A and one in B. Therefore (A, B) is a bipartition of Q_n, completing the proof. □

Question:	What is the relationship between this bipartition and distances in Q_n?
Answer:	This definition places a vertex in A whenever it is at an even distance from vertex $00 \ldots 0$.

Proposition 12.3. *All trees are bipartite.*

Proof. We know from Theorem 10.2 that trees have no cycles at all—in particular, they have no odd cycles, so they are bipartite by Theorem 12.1. □

In Chapter 10, we already proved an equivalent statement to Proposition 12.3: it was Proposition 10.7. Our proof back then was much longer, because we did not have Theorem 12.1 at our disposal!

12.4 Matching problems

Generalizing the rook placement puzzle, we make the following definition.

Definition 12.3. *A **matching** in a graph G is a set of edges $M \subseteq E(G)$ such that no two edges in M share an endpoint. We can also view M as a spanning subgraph of G in which every vertex of G has degree 0 or 1.*

It will be convenient to go back and forth between the two versions of the definition: as a set of edges, and as a spanning subgraph.

We can think about matchings in all kinds of graphs. But we will start by thinking about matchings in bipartite graphs for two reasons. First, many applications (like the rook placement puzzle) will naturally give us a bipartite graph in which to look for a matching. Second, the theory turns out to be simpler for bipartite graphs.

As we've phrased this definition, the empty set (or the subgraph containing all vertices of G, but no edges) is also a matching—it's just not a very good one. It is more difficult, but also more interesting, to select more edges. The limit on selecting more edges is that vertices get "used up", for which we have the following terminology.

Definition 12.4. *A vertex x is said to be **covered by M** if x is the endpoint of some edge of M, and **uncovered** otherwise. Viewing M as a subgraph, x is covered by M if $\deg_M(x) = 1$, and uncovered by M if $\deg_M(x) = 0$.*

The larger a matching is, the more vertices it covers, and so the absolute limit is to cover every vertex of G.

Definition 12.5. *A matching M in a graph G is **perfect** if it covers every vertex of G.*

The bipartite matching problem can be considered from two points of view. The first is the optimistic point of view, in which we ask whether a perfect matching exists. This is very common in theoretical applications, including applications to other areas of math. For example, when we ask "can we place a rook in every rank and every file of the chessboard?" then we are looking for a perfect matching.

There is also what I call the "realistic" point of view. In many practical applications, while a perfect matching is still the best option, a very big matching is still good if it is not perfect. For example, one practical bipartite matching problem occurs in every math department when instructors are assigned to courses. It might not be possible in a given semester to offer every course, but does the math department give up and cancel classes for the semester? No! Instead, the department simply tries to schedule as many courses as possible.

Question:	What bipartite graph can we use to model assigning instructors to courses?
Answer:	The vertices of the graph are the course sections and the instructors that can teach them; the edges represent which courses an instructor is qualified to teach.

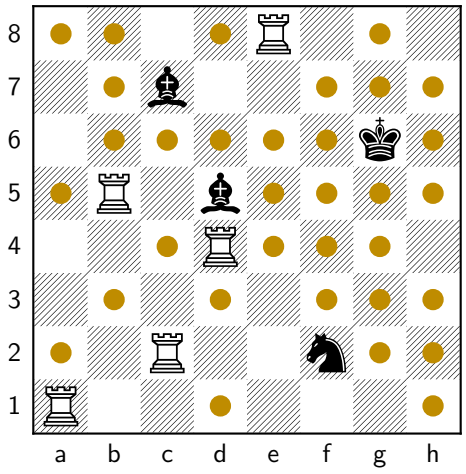


Figure 12.3: An inefficient but maximal placement of rooks

(Further complications on top of the matching problem arise when we consider giving an instructor multiple sections, which must not be scheduled at the same time.)

In general, instead of asking whether a perfect matching exists, the second point of view is to ask how big the largest matchings are. It's this point of view that we will start with in this textbook. Of course, a complete answer to this question will also tell us whether a perfect matching exists!

12.5 Maximum and maximal

In combinatorial optimization problems like the matching problem, we are trying to find a set S that's as large as possible, subject to some condition. It's common to say that a set S is:

- **maximum** if it really is as large as you can get; there are no better solutions than S.
- **maximal** if you cannot add anything to S without violating the restriction; there might be better solutions, but you cannot get to them without removing something from S first.

Every maximum solution is also maximal, but the reverse might not hold.

In the same way, the words **minimum** and **minimal** get used when we're trying to pick a set that's as small as possible, subject to some condition.

In this textbook, I will also use this terminology, but I will avoid conveying important information solely by the difference between "maximum" and "maximal". Usually, we are interested in maximum and minimum objects, because we want to solve the optimization problems we pose, and I will let this situation pass by without further comment. If we really need a maximal object that is not necessarily maximum, or a minimal object that is not necessarily minimum, I will specifically point this out for you.

However, no matter which terminology you use, it's important to realize that there *is* a difference, and in particular there is a difference in the matching problem. For example, consider

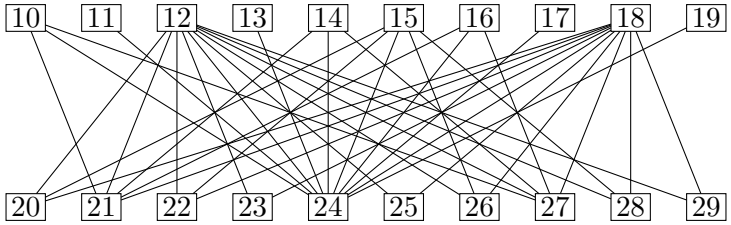


Figure 12.4: The multiples-of-six graph

the rook placement in Figure 12.3, which I have specifically designed to be as poor an attempt as possible to solve the problem. It is a failed attempt, in that we have not placed a rook in every rank and every file. However, no more rooks can be placed: there is no unoccupied square not attacked by any black pieces where we can add a rook. So this is a maximal placement of rooks.

It is not a maximum placement, however! For example, if we remove the rook on square `a1` (the bottom left corner), then this lets us place two rooks in exchange: one above it on square `a6` and one to its right on square `g1`.

This example also shows us that the bipartite matching problem cannot be solved by a greedy algorithm that selects the first edges it sees. We should expect to have to work hard before we find a general solution.

12.6 Vertex covers

Here is a new, more abstract example of a bipartite matching problem. Let G be the bipartite graph with vertices $\{10, 11, \ldots, 19\}$ on one side, $\{20, 21, \ldots, 29\}$ on the other side, and edges defined by the following rule: vertices x and y are adjacent when they begin with different digits and the product $x \cdot y$ is divisible by 6. A diagram of the multiples-of-six graph is shown in Figure 12.4, though it is a little busy in places. What is the largest matching in this graph?

Question:	Is there a perfect matching in this graph?
Answer:	No: vertices 11 and 13 on the top are only adjacent to vertex 24, so they cannot both be covered by a matching.

Question:	Look for a matching greedily: for each of the vertices $10, 11, \ldots, 19$, match it to the first unused vertex on the other side, if you can. What matching do you get?
Answer:	The matching $\{10, 21\}, \{11, 24\}, \{12, 20\}, \{14, 27\}, \{15, 22\}, \{18, 23\}$.

We've made a good start, but we have a gap: we've found a matching with 6 edges, but we can only prove that the matching can't have more than 9 edges. (That is, we've ruled out a perfect matching with 10 edges.) We either need to find a better solution, or we need to prove a better

upper bound. Is there a bottleneck—some kind of limited resource that we exhaust if we try to find a larger matching?

Yes! The limited resource is the vertices divisible by 3. To get a product divisible by 6, we need both a multiple of 2 and a multiple of 3, but the multiples of 3 are much harder to come by: they are $12, 15, 18, 21, 24$, and 27. Every edge in a matching needs to use one of these vertices, so there can be at most 6 edges in a matching.

This argument generalizes.

Definition 12.6. *A **vertex cover** in a graph G is a set of vertices $U \subseteq V(G)$ such that every edge of G has one or both endpoints in U.*

For example, in the multiples-of-six graph, the set $\{12, 15, 18, 21, 24, 27\}$ is a vertex cover. It is not the only one. For example, the set of *all* vertices in a graph is guaranteed to be a vertex cover. The hard task is not to find a vertex cover, but to find a vertex cover that is as small as possible.

Vertex covers are important because, just as in the example of the multiples-of-six graph, we can use them to prove upper bounds on the size of a matching:

Proposition 12.4. *If M is a matching and U is a vertex cover in the same graph G, then $|M| \leq |U|$.*

Proof. The quantities $|M|$ and $|U|$ are interpreted by treating M and U as sets; however, the proof is simpler to state if we think of M as a subgraph of G, rather than a set of edges. Consider the sum

$$\sum_{x \in U} \deg_M(u).$$

Each term $\deg_M(x)$ of this sum is at most 1, by the definition of a matching, so the value of the sum is at most $|U|$: the number of vertices in U.

However, $\deg_M(u)$ counts the number of edges of M incident on x, so we can also think of the sum as counting the number of times an edge of M is incident on any vertex of U. Each edge of M must be incident on at least one vertex of U, by the definition of a vertex cover, so each edge of M contributes at least 1 to the sum. Therefore the sum is at least the number of edges in M.

Putting these inequalities together, we conclude that $|M| \leq |U|$. □

Question:	In a graph with bipartition (A, B), there are two sets that are guaranteed to be vertex covers. What are they?
Answer:	The sets A and B are vertex covers; this follows almost immediately from the definition of a bipartition.

Question:	Does Proposition 12.4 only apply to bipartite graphs?
Answer:	No: it is true for any graph.

Proposition 12.4 can be used with any matching M and any vertex cover U. However, if we want to get upper bounds on the size of a matching, then the smaller U is, the better the bounds we get. To get the best upper bound, we should try to find the smallest vertex cover possible.

In general, even for the smallest vertex cover in a graph, this bound might not tell us the full truth about matchings.

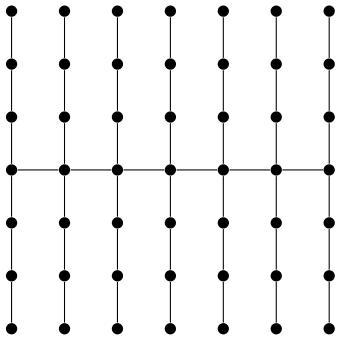
Question:	In the complete graph K_{100}, how many edges are in the largest matching?
Answer:	There are 50, because a 50-edge matching covers all 100 vertices: it is perfect.

Question:	In the complete graph K_{100}, how many vertices are in the smallest vertex cover?
Answer:	There are 99. We can leave out any one vertex from the vertex cover, but a set U missing two vertices x and y does not contain either endpoint of the edge xy.

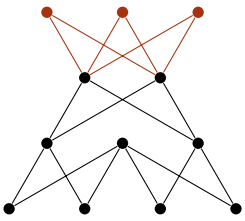
It will turn out that in bipartite graphs, the problems of finding the largest matching and the smallest vertex cover do "meet in the middle". This is a result known as Kőnig's theorem, which we will prove in the next chapter. It is the reason we are studying matchings in bipartite graphs first.

12.7 Practice problems

1. In the graph shown below, find a matching M and a vertex cover U with $|M| = |U|$.



2. In the "volcano graph" shown below, find a bipartition (A, B). Then, find a vertex cover showing that no perfect matching exists.



3. Mathematicians that study arrangements of non-attacking rooks on chessboards of various shapes often define an object called the **rook polynomial**. This is a polynomial $p(x)$ in which the coefficient of x^k is the number of ways to place k non-attacking rooks onto the chessboard.

   a) Find the rook polynomial of a 5×5 chessboard.

   b) Find the rook polynomial of a 5×5 chessboard with one square missing.

4. The "cop-and-highway-robber game"[3] is played as follows.

   The playing board is a bipartite graph. One player (the highway robber) chooses an edge of the graph and occupies it for highway robbery. Simultaneously and without seeing the edge chosen, the other player (the cop) chooses a vertex of that edge to guard. If the cop guards an endpoint of the edge chosen by the highway robber, the highway robber gets caught and loses; otherwise, the highway robber gets away and wins.

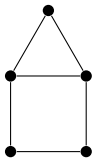
   This is a game, like rock-paper-scissors, where both players need to choose their strategy at random; if either player plays predictably, then the other player can exploit this.

   a) Suppose that the graph has a matching M. Find a strategy for the highway robber to win with probability $1 - \frac{1}{|M|}$, no matter what the cop does (and prove that it works).

   b) Suppose that the graph has a vertex cover U. Find a strategy for the cop to win with probability $\frac{1}{|U|}$, no matter what the highway robber does (and prove that it works).

      (Why is this another proof of Proposition 12.4?)

   c) If $|M| = |U|$, these two strategies "meet in the middle" and end up being both optimal. If there is no M and U such that $|M| = |U|$, then it's possible that neither strategy is optimal. For an example of this, see if you can figure out the optimal strategies for both players on the "house graph" shown below:



5. To explore the difference between maximum and maximal matchings, consider the following graph (which can be made arbitrarily large):

   a) Find a perfect matching in this graph.

   b) Find the smallest matching in this graph above which is maximal (there is no larger matching that contains all of its edges).

[3] Whose name I made up, inspired by the study of cops-and-robbers games on graphs, which are something fancier.

c) Let G be a graph (not necessarily bipartite) and let M be a maximal matching in G. Prove that G has a vertex cover U with $|U| = 2|M|$.

d) Prove that the example in this problem is essentially as bad as it gets: if M is a maximal matching, then there is no matching with more than $2|M|$ edges.

13 Kőnig's theorem

The purpose of this chapter

This chapter presents the algorithmic view of matchings in bipartite graphs. The augmenting path algorithm is the most serious algorithm seen in this textbook so far; accordingly, I've given it the most serious treatment, with a detailed proof that the algorithm works correctly, and an example of how the algorithm is used. I do not intend to treat every algorithm similarly; this is not a computer science textbook. This *is*, however, an introductory graph theory textbook, so I hope that this chapter introduces you to the algorithmic side of graph theory.

In my opinion, Kőnig's theorem (Theorem 13.2) is the right setting to consider the algorithmic questions, because it is a result comparing two optimization problems, which is how we usually encounter matchings in algorithmic applications. Of course, Kőnig's theorem has its uses in pure mathematics, and Hall's theorem (Theorem 14.1 in the next chapter) can also be proved by giving an algorithm. However, most uses of Hall's theorem are about abstractly proving the existence of a perfect matching in a general, incompletely specified family of graphs—and that is the sort of problem we will consider in the next chapter.

13.1 Kőnig's theorem

As we've already seen in Chapter 4 with the notation for minimum degree ($\delta(G)$) and maximum degree ($\Delta(G)$), Graph theorists like to use Greek letters for numerical invariants of graphs. We have recently learned two new such invariants, so let's learn the notation for them.

Definition 13.1. *The **matching number** of a graph G, denoted $\alpha'(G)$, is the number of edges in a maximum (largest) matching of G.*

Definition 13.2. *The **vertex cover number** of a graph G, denoted $\beta(G)$, is the number of vertices in a minimum (smallest) vertex cover of G.*

Unfortunately, the letters α ("alpha") and β ("beta") here have no particular meaning to help you remember them. There *is* a meaning to the apostrophe present in $\alpha'(G)$ but not in $\beta(G)$: it indicates that the invariant is an "edge version" of another invariant. This means that you can already begin to guess what $\alpha(G)$ (the "vertex version" of the matching number) might mean: later on, in Chapter 17, you will learn if your guess was right.

Using the new notation, we can rewrite Proposition 12.4, which we stated as an inequality between $|M|$ (the number of edges in a matching M) and $|U|$ (the number of vertices in a vertex cover U). The new version reads:

Proposition 13.1. *In any graph G, $\alpha'(G) \leq \beta(G)$.*

Actually, it might not be immediately obvious that Proposition 12.4 and Proposition 13.1 are the same: the first is talking about an arbitrary matching and vertex cover, and the second is only talking about maximum matchings and minimum vertex covers.

However, if Proposition 12.4 holds for all M and U, then in particular it holds when M is a maximum matching and U is a minimum vertex cover, which proves Proposition 13.1. Conversely, if Proposition 13.1 holds, then for any matching M and vertex cover U, we have $|M| \leq \alpha'(G) \leq \beta(G) \leq |U|$, proving Proposition 12.4. (The inequalities $|M| \leq \alpha'(G)$ and $\beta(G) \leq |U|$ come from the definitions of "maximum" and "minimum".)

The main theorem of this chapter was proved in 1931 by Dénes Kőnig [4]. Actually, it is ambiguous which result you mean if you say "Kőnig's theorem", because Kőnig proved many results in graph theory: he was, in fact, the author of the first graph theory textbook [5], written in 1936.[1] In this textbook, the theorem we'll call Kőnig's theorem is the following:

Theorem 13.2 (Kőnig's theorem)**.** *For any bipartite graph G, $\alpha'(G) = \beta(G)$.*

To prove this theorem, we will find an algorithm that does the following. It starts with any matching M, and then either constructs a larger matching M', or find a vertex cover U with $|M| = |U|$.

Question:	If we find such a U, what can we conclude?
Answer:	Since every matching must be at most as big as U, and M has the same size as U, we know that M is a maximum matching by Proposition 12.4. Similarly, U is a minimum vertex cover.

However, we know that we cannot always improve a suboptimal matching simply by adding edges. So the question we'll look at next is: what *does* it take to improve a suboptimal matching?

13.2 Augmenting paths

We will need a set-theoretic operation that is not as well-known as union and intersection, but is the natural operation to consider when tracking changes. (This remains true whether we're tracking changes made to a matching in a graph, or tracking changes made to a Wikipedia entry, for example.) At its most basic, it's defined on sets: if A and B are two sets, then their **symmetric difference $A \oplus B$** is the set $(A \cup B) - (A \cap B)$: the set containing all elements in A, or in B, but not in both A and B.

We can think of $A \oplus B$ as a summary of the changes that need to be made to A to get B. If we start with A, and "toggle" all the elements of $A \oplus B$ (removing them from A if they are in A, and adding them to A if not), then the result is B.

[1] By a happy coincidence, this is 200 years after Euler first modeled a problem—the problem of the seven bridges of Kőnigsberg—using what we'd now call a graph.

IMO 2024	IMO 2025	2024 ⊕ 2025
United States of America	People's Republic of China	Belarus
People's Republic of China	United States of America	United Kingdom
Republic of Korea	Republic of Korea	Hungary
India	Japan	Japan
Belarus	Poland	Israel
Singapore	Israel	Vietnam
United Kingdom	India	
Hungary	Singapore	
Poland	Vietnam	
Türkiye	Türkiye	

Figure 13.1: Top ten IMO teams in 2024 and in 2025

Question: If we know A and $A \oplus B$, what operation on them lets us find B?

Answer: It's symmetric difference again: B is $A \oplus (A \oplus B)$. The elements in $A \oplus B$ but not A are the elements only in B, which we need to add to A; the elements in both A and $A \oplus B$ are exactly the ones not in B, which we need to remove.

Question: Which elements does a triple symmetric difference $(A \oplus B) \oplus C$ contain?

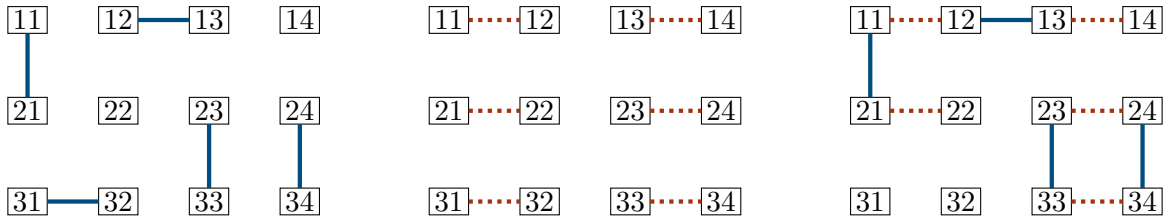
Answer: The elements contained in an odd number (one, or all three) of the sets A, B, C. This rule generalizes, and can also be used to prove that $\oplus$ is associative: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, because the rule applies to both sides.

For example, Figure 13.1 shows a table listing the top ten countries in the International Mathematical Olympiad in 2024 and in 2025. If we think of these lists as sets, we can take the symmetric difference, shown in the third column. This symmetric difference shows the six countries whose status changed from 2024 to 2025: Belarus, the United Kingdom, and Hungary left the top ten list, while Japan, Israel, and Vietnam entered it.

Rather than continue thinking about competition performance, let's use this idea to see what changes need to be made to get from one matching to another. We are looking for small, incremental changes. So we will compare two matchings M_1 and M_2 such that M_2 only has one more edge than M_1.

Figure 13.2 shows one such example. To make the drawing more clear, I have not included the edges of the original graph (which you may assume to be a 3×4 **grid graph**, like the grid graphs we saw in Chapter 10). Matching M_1 (in Figure 13.2a) has only 5 edges, while M_2 (in Figure 13.2b) is a perfect matching with 6 edges.

As a graph, $M_1 \oplus M_2$ (shown in Figure 13.2c) consists of four connected components. Two of them are just isolated vertices, and don't have a lot to tell us. Another, with the vertices $\{23, 24, 33, 34\}$, is a cycle, and represents only that the matchings M_1 and M_2 match those four vertices in two different ways.



(a) Matching M_1 (b) Matching M_2 (c) $M_1 \oplus M_2$

Figure 13.2: Two matchings in a 3×4 grid

The last component is the most interesting one. It is isomorphic to a path graph, and the path through its vertices is the path $(22, 21, 11, 12, 13, 14)$. It's this component that tells us how to improve matching M_1: we must swap the two edges $\{11, 21\}$ and $\{12, 13\}$ for the three edges $\{11, 12\}$, $\{13, 14\}$, and $\{21, 22\}$.

In general, the structure of such a component is as follows:

Definition 13.3. *If M is a matching in a graph G, then an **M-augmenting path** is a path in G with the following properties:*

- *The path begins and ends at two vertices that are not covered by M.*
- *The edges used by the path alternate between edges not in M and edges in M.*

*A path that has the second property, but not necessarily the first, is called an **M-alternating path**.*

In order for both properties of an M-augmenting path to hold, the path needs to have odd length. If it has length $2k + 1$, then the $1^{\text{st}}, 3^{\text{rd}}, \ldots, (2k + 1)^{\text{th}}$ edges are not part of M, while the $2^{\text{nd}}, 4^{\text{th}}, \ldots, (2k)^{\text{th}}$ edges are.

Question:	Can a cycle in $M_1 \oplus M_2$ ever have more edges from one matching than the other?
Answer:	No: the edges on the cycle must alternate between edges of M_1 and edges of M_2, because a vertex cannot be the endpoint of two edges in the same matching. This forces the cycle to have an even number of edges, half of which are in M_1 and half in M_2.

Question:	Apart from augmenting paths, isolated vertices, and cycles, can $M_1 \oplus M_2$ have any other components?
Answer:	There is one more type we have not seem: paths of even length with the same number of edges from each matching. These are not augmenting paths, but they are still alternating paths.

Lemma 13.3. *Let M be a matching in a graph G and let P be the set of edges on an M-augmenting path $(x_0, x_1, \ldots, x_{2k+1})$. Then $M \oplus P$ is a matching with one more edge than M.*

Proof. First, we check that $M \oplus P$ is a matching: that every vertex of G has degree at most 1 in $M \oplus P$.

Vertices x_0 and x_{2k+1} are uncovered by M: they have degree 0 in M. Each is incident on just one edge of P: x_0x_1 and $x_{2k}x_{2k+1}$, respectively. These edges are also in $M \oplus P$, giving these vertices degree 1.

Every other vertex x_i on the path is incident on two edges of P: $x_{i-1}x_i$ and x_ix_{i+1}. One of these is in M and the other is not, so in $M \oplus P$ we remove one edge and add the other; x_i still has degree 1 in $M \oplus P$.

Finally, all vertices not on the path are unaffected by the change from M to $M \oplus P$: they still have the same degree they had in M, which is at most 1. Therefore $M \oplus P$ is a matching.

It is a matching with one more edge than P because P contains $k+1$ edges not in M (edges of the form $x_{2i}x_{2i+1}$ for $i = 0, 1, \ldots, k$) but only k edges in M (edges of the form $x_{2i-1}x_{2i}$ for $i = 1, \ldots, k$). The $k+1$ edges are added and the k edges are removed, for a net gain of 1 edge. □

Our classification of the connected components in $M_1 \oplus M_2$, the symmetric difference of two matchings, proves a converse to Lemma 13.3. Suppose M is a matching which is *not* a maximum matching: there is a bigger matching N. If M and N have k edges in common, then $M \oplus N$ contains $|M| - k$ edges from M and $|N| - k$ edges from N: at least one more edge from N than M.

Question:	How can a component of $M \oplus N$ contain more edges from N than M?
Answer:	Only if it is an alternating path of odd length where the first and last edge both come from N. This is an M-augmenting path!

In particular, this argument proves that there exists an M-augmenting path, but not usefully: to find it, we needed to know the bigger matching N. In our proof of Kőnig's theorem, we will see how to find an M-augmenting path in a more helpful fashion, but only in bipartite graphs.

13.3 The augmenting path algorithm

Let G be a bipartite graph with bipartition (A, B), and let M be a matching in G. The main tool we need to prove Theorem 13.2 is an algorithm for finding an M-augmenting path.

Our algorithm will be an exploration algorithm, like the distance-finding algorithm in Chapter 3 or like the algorithm for finding a bipartition in Chapter 12. However, we will not explore along all possible trajectories, but only among ones that can form an M-augmenting path.

We are searching for a path between two uncovered vertices, so let's introduce some notation to help us. Write A as $A_0 \cup A_1$, where the vertices in A_0 are uncovered by M and the vertices in A_1 are covered by M. (That is, $x \in A_k$ if x has degree k in M.) Similarly, write B as $B_0 \cup B_1$.

Since an augmenting path has odd length, and every path in G alternates between A and B, an augmenting path must have one endpoint in A_0 and one endpoint in B_0. We will arbitrarily choose to start our exploration in A_0, with the goal of eventually reaching a vertex in B_0.

To keep track of our progress, let A_{exp} and B_{exp} be the set of **exp**lored vertices in A and in B, respectively. These will change over the course of the algorithm. Since we start our exploration in A_0, initially $A_{\text{exp}} = A_0$ and $B_{\text{exp}} = \varnothing$.

Finally, we will also want to know *how* we reached the explored vertices: when we add a new vertex x to $A_{\text{exp}} \cup B_{\text{exp}}$, we make note of $f(x)$, the vertex **f**rom which we explored x. This will help us find an augmenting path, if one exists.

The augmenting path algorithm repeats the following steps, which I've given short names to help us refer to them.

1. **Explore:** For each vertex $a \in A_{\text{exp}}$, explore all its neighbors: if a neighbor b of a is not already in B_{exp}, add it, and set $f(b) = a$. *(After the first iteration, this only needs to be done for the vertices newly added to B_{exp}.)*

2. **Check:** At this step, there are two stopping conditions to check, which conclude the algorithm with different results.

   - **Path?** If there is a vertex $b \in B_{\text{exp}} \cap B_0$ (an explored vertex uncovered by M), stop and return the path
   $$(b, f(b), f(f(b)), \dots)$$
   that traces back the vertices from which we reached b, ending at a vertex in A_0. We will show that this is an M-augmenting path.

   - **Cover?** If no new vertices were added to B_{exp} in the most recent **Explore** step, stop and return the set U defined[2] to be $(A - A_{\text{exp}}) \cup B_{\text{exp}}$. We will show that U is a vertex cover and $|U| = |M|$.

3. **Match:** Otherwise, $B_{\text{exp}} \subseteq B_1$: all the vertices in B_{exp} are covered by M. For each vertex b added to B_{exp} in the most recent **Explore** step, find the edge $ab \in M$ incident on b. Explore vertex a (the vertex that b is matched to): add it to A_{exp}, and set $f(a) = b$.

To prove the correctness of this algorithm, we must show that when the **Path** or **Cover** stopping conditions are met, the path really is an M-augmenting path and the cover really is a vertex cover of the same size as M.

Question:	We must also show that the algorithm cannot loop forever. Why must it halt?
Answer:	At every iteration in which we don't stop by the **Cover** condition, we add a new vertex to B_{exp}, and $\lvert B_{\text{exp}}\rvert$ is limited in size by $\lvert B\rvert$.

In practice, this algorithm is applied multiple times. Starting from an initial matching, which does not need to be very good, we repeatedly use the algorithm to grow the matching by one edge. Eventually, we either find a perfect matching, or find a vertex cover that proves we cannot proceed any further. This self-certifying feature, in addition to being theoretically useful in the proof of Theorem 13.2, is also practically convenient: it means that we can verify the output of the algorithm independently if we're not certain that an implementation of it is bug-free.

Which initial matching do we start with? One option is to begin with $M = \varnothing$.

[2] If you're learning about this algorithm for the first time, this definition may feel like it came out of nowhere! In the next section, we'll discuss where this formula for the vertex cover comes from.

Question:	What does the algorithm do if $M = \varnothing$?
Answer:	We start with all vertices of A already in A_{exp}, and all vertices of B in B_0. If even a single edge exists, we will explore a vertex of B_0 and halt in the first step, returning a path of length 1.

Question:	When a path (a, b) of length 1 is an M-augmenting path, what does this mean?
Answer:	It means that the edge ab can be added to M without removing any edges.

Starting from $M = \varnothing$, not just the first time we augment the matching but the first few times will typically end equally quickly: they will find an edge between two uncovered vertices. In effect, this is no different from the greedy algorithm. Only once the greedy algorithm would give up, because it cannot add any more edges, does the augmenting path algorithm start finding longer augmenting paths.

13.4 Analyzing the algorithm

A common technique when analyzing an algorithm is to prove an **invariant** of the algorithm. Just like a graph invariant is a property that does not change when the graph is relabeled, an invariant of an algorithm is also a property that does not change—it does not change over the course of the algorithm.

There are two invariants we track for the augmenting path algorithm: one invariant to help us find an M-augmenting path, and one invariant that

For the first invariant, we define a path $P(x)$ for each vertex $x \in A_{\text{exp}} \cup B_{\text{exp}}$. It is the path

$$(x, f(x), f(f(x)), \dots)$$

that starts at x and traces back the vertices from which we reached x, ending at a vertex in A_0. Recall that in the **Path** stopping condition, this is the path we return, for a vertex $x \in B_{\text{exp}} \cap B_0$.

Lemma 13.4. *For any vertex $x \in A_{\text{exp}} \cup B_{\text{exp}}$, the path $P(x)$ is an M-alternating path.*

Proof. More precisely, we prove the following: an edge $yf(y)$ used by $P(x)$ is in M if $y \in A_{\text{exp}}$, and not in M if $y \in B_{\text{exp}}$. The path $P(x)$ must alternate between A_{exp} and B_{exp}: like any other path in G, it alternates between A and B, and it only passes through explored vertices. So this claim will prove that $P(x)$ is an M-alternating path.

Let $a \in A_{\text{exp}}$ be a vertex on $P(x)$. Then either $a \in A_0$ (in which case, it is the end of the path) or else a was explored from a vertex $f(a) \in B_{\text{exp}}$. We explore from vertices in B_{exp} in a **Match** step; therefore $af(a) \in M$.

Let $b \in B_{\text{exp}}$ be a vertex on $P(x)$. Then b was explored from a vertex $f(b) \in A_{\text{exp}}$; we must prove that $bf(b) \notin M$. If $f(b) \in A_0$, this is automatic, because then M has no edges incident on M. Otherwise, $f(b)$ was itself explored from some vertex $f(f(b))$, and as we've proved above, $f(b)f(f(b)) \in M$. Therefore $bf(b) \notin M$, because $f(b)$ cannot be incident on two edges of M. This completes the proof. □

Next, we turn our attention to the set $U = (A - A_{\text{exp}}) \cup B_{\text{exp}}$, which we return if the **Cover** stopping condition is satisfied. First of all, let's understand where this set comes from.

Question:	If a vertex cover U satisfies $\lvert U\rvert = \lvert M\rvert$, can U ever use both endpoints of an edge in M?
Answer:	No: if we select just one endpoint of each edge in M, we've already selected $\lvert M\rvert$ different vertices. Choosing both endpoints of an edge in M is wasteful and we cannot afford it.

Question:	If a vertex cover U satisfies $\lvert U\rvert = \lvert M\rvert$, how must it treat A_0 and B_0?
Answer:	U cannot contain any vertices in A_0 and B_0. Again, we've already reached $\lvert M\rvert$ vertices just by selecting a vertex from each edge in M. We can't select any more without going over our limit, so we can't select any vertices not covered by M.

These two guiding questions tell us what to do to the vertices we explore. Our initial set A_{exp} consists just of A_0: these vertices cannot be part of U. Then, to cover the edges incident on A_0, we must add all their neighbors to U. These neighbors are exactly what we put in B_{exp} in the **Explore** step. Next, if a vertex b we've added to U is an endpoint of an edge $ab \in M$, we can't put the other endpoint of ab in U as well, so $a \notin U$. So the vertices matched to B_{exp} cannot be part of U, and these are exactly the vertices we put in A_{exp} in the **Match** step.

This logic continues: at each step, vertices we put in A_{exp} are vertices we know cannot be part of U, and vertices we put in B_{exp} are vertices we know must be part of U. Since we don't know anything about vertices outside $A_{\text{exp}} \cup B_{\text{exp}}$ a reasonable first try is to define $U = (A - A_{\text{exp}}) \cup B_{\text{exp}}$, and we we will see that this turns out to work.

We can define $U = (A - A_{\text{exp}}) \cup B_{\text{exp}}$ at any point over the course of the algorithm, not just at the end. It is not a vertex cover before the end, but it satisfies an invariant of its own:

Lemma 13.5. *Immediately before every **Explore** step, $\lvert U\rvert = \lvert M\rvert$.*

Proof. At the beginning of the algorithm, $A_{\text{exp}} = A_0$ and $B_{\text{exp}} = \varnothing$; therefore $U = A - A_0 = A_1$. Every edge of M has one endpoint in A_1 and one endpoint in B_1; therefore $\lvert U\rvert = \lvert A_1\rvert = \lvert M\rvert$.

In the **Explore** step, some vertices are added to B_{exp}. Then, in the **Match** step, for every vertex b that we added to B_{exp}, we add a vertex a to A_{exp}: the vertex a such that $ab \in M$. Such a vertex a cannot already have been in A_{exp}: it is not in A_0 (because $ab \in M$), and so it can only be explored in the **Match** step, and only from b.

Therefore, if we add k vertices to B_{exp} in the **Explore** step, we also add k vertices to A_{exp} in the **Match** step (provided we do not stop before then). As a result, $|A - A_{\text{exp}}|$ decreases by k and $|B_{\text{exp}}|$ increases by k, so $|U| = |A - A_{\text{exp}}| + |B_{\text{exp}}|$ remains unchanged

Since $|U|$ starts equal to $|M|$, and a single **Explore**–**Check**–**Match** iteration of the algorithm does not change $|U|$, we must also have $|U| = |M|$ before every **Explore** step. □

Many uses of invariants to study algorithms are proofs by induction in disguise, and you can see that in the proof of Lemma 13.5. Here, we begin with a base case: we show that $|U| = |M|$ at the beginning of the algorithm. Next, we show that whenever this property holds, it still holds after one more iteration.

Together, Lemma 13.4 and Lemma 13.5 are not enough to draw any conclusions about what happens at the end of the algorithm: we need to see how the stopping conditions contribute.

Proposition 13.6. *If the **Path** stopping condition holds, then for a vertex $b \in B_{\text{exp}} \cap B_0$, the path $P(b)$ is an M-augmenting path.*

Proof. We already know from Lemma 13.4 that $P(b)$ is an M-alternating path. Moreover, its endpoints are uncovered by M: its start is b, which is in B_0, and its end can only be a vertex in A_0, because all other explored vertices were explored from somewhere. Therefore $P(b)$ is an M-augmenting path. □

Proposition 13.7. *If the **Cover** stopping condition holds, then $U = (A - A_{\text{exp}}) \cup (B_{\text{exp}})$ is a vertex cover with $|M| = |U|$.*

Proof. To show that U is a vertex cover, take any edge $ab \in E(G)$, where $a \in A$ and $b \in B$. If $a \in A_{\text{exp}}$, then in the most recent explore step, we would have made sure that $b \in B_{\text{exp}}$ (if it was not already there for some other reason); in this case, $b \in U$. But if $a \notin A_{\text{exp}}$, then $a \in U$. In either case, one of the endpoints of ab is in U.

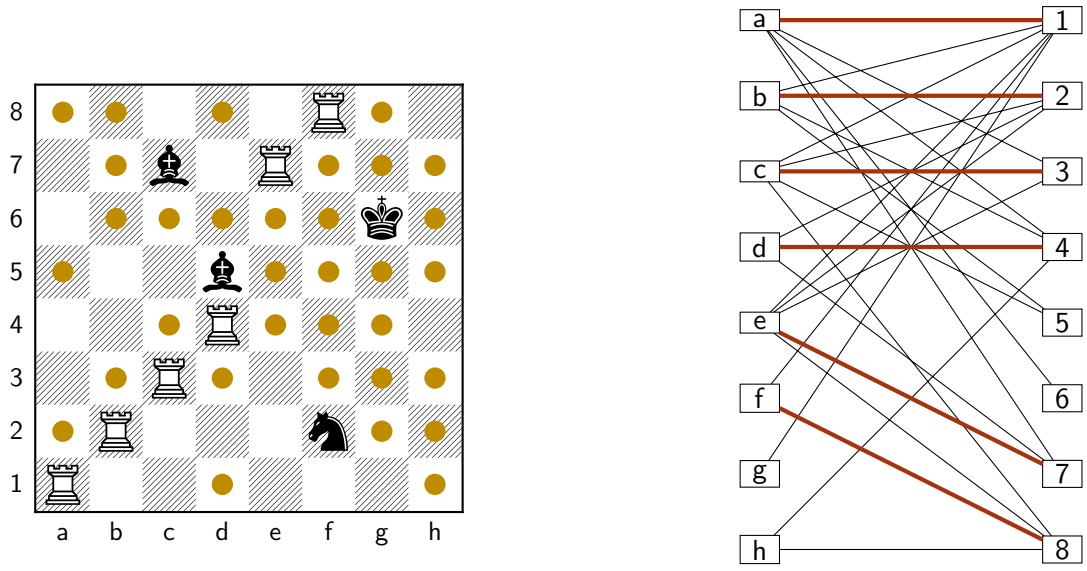
From Lemma 13.5, we know that $|M| = |U|$ before any **Explore** step. If the **Cover** stopping condition holds, then no new vertices were added to A_{exp} in the explore step, so $|U|$ did not change; therefore $|M| = |U|$ at the end of the algorithm. □

This completes the proof that the algorithm is correct. It also gives us all the tools we need to prove Kőnig's theorem.

Proof of Theorem 13.2. Let M be a maximum matching in the bipartite graph G; starting with the matching M, run the augmenting path algorithm.

The algorithm cannot find an M-augmenting path, because by Lemma 13.3, we could use it to obtain a bigger matching, which contradicts our choice of M. So it must stop by satisfying the **Cover** stopping condition. By Prop 13.7, the algorithm outputs a vertex cover U with $|M| = |U|$.

Since $\alpha'(G) = |M|$ (by our choice of M) and $\beta(G) \leq |U|$ (the minimum vertex cover is at least as small as U), we conclude that $\alpha'(G) \geq \beta(G)$. From Proposition 13.1, we know $\alpha'(G) \leq \beta(G)$; therefore the two are equal. □



(a) Six greedily placed rooks

(b) $M = \{\mathsf{a1}, \mathsf{b2}, \mathsf{c3}, \mathsf{d4}, \mathsf{e7}, \mathsf{f8}\}$

Figure 13.3: An initial greedy 6-rook placement

13.5 An example of using the algorithm

Algorithms like the augmenting path algorithm are not really intended to be performed by hand. Very rarely, if a graph is neither so small that we can find a matching without any algorithms, nor too big to deal with by hand, we can use the algorithm to check a matching we suspect to be optimal. Most of the time, though, you should use a computer.

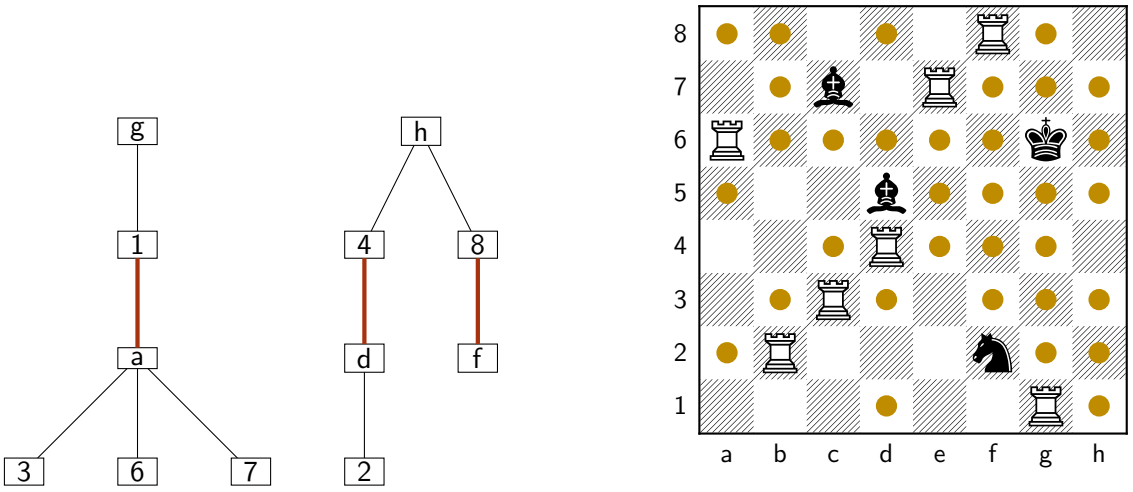
In this section, we will apply the augmenting path algorithm to solve the rook placement problem from the previous chapter. This is intended as an illustration, so that you can better understand the algorithm and its proof by seeing it in action. (You will gain even better understanding if you try doing this yourself, and I've included a practice problem about this. After you understand the algorithm perfectly, there's no point.)

I have mentioned that the first few rounds of finding augmenting paths are essentially equivalent to the greedy algorithm, so I will skip them to avoid wasting time on cases that don't help us understand anything. We will begin with the rook placement in Figure 13.3a, which corresponds to the matching highlighted in Figure 13.3b.

We must make an arbitrary choice between the two sides: let's say that side A (the side that we explore from) is $\{\mathsf{a}, \mathsf{b}, \ldots, \mathsf{h}\}$ and side B is $\{1, 2, \ldots, 8\}$. (In chess, squares are labeled as $\mathsf{a1}$ rather than $\mathsf{1a}$, so this is the natural choice.) Given the matching M in Figure 13.3b, we start with $A_0 = \{\mathsf{g}, \mathsf{h}\}$ and $B_0 = \{5, 6\}$.

The algorithm proceeds as follows:

1. **Explore** the vertices 1, 4, and 8, adding them to B_{exp}, with $f(1) = \mathsf{g}$, and $f(4) = f(8) = \mathsf{h}$.
2. **Check** the stopping conditions:
   - Is there a **Path?** No: $B_{\text{exp}} \cap B_0 = \varnothing$.



(a) The exploration forest

(b) Rooks on $\{\mathsf{a6}, \mathsf{b2}, \mathsf{c3}, \mathsf{d4}, \mathsf{e7}, \mathsf{f8}, \mathsf{g1}\}$

Figure 13.4: From 6 rooks to 7 rooks

- Is there a **Cover?** No: we added 3 new vertices to B_{exp}.

3. **Match** vertex 1 to a, vertex 4 to d, and vertex 8 to f. Add a, d, and f to A_{exp}, with $f(\mathsf{a}) = 1$, $f(\mathsf{d}) = 4$, and $f(\mathsf{f}) = 8$.
4. **Explore** the vertices 2, 3, 6, and 7, with $f(3) = f(6) = f(7) = \mathsf{a}$ and $f(2) = \mathsf{d}$. As before, add them to B_{exp}.
5. **Check** the stopping conditions:
   - Is there a **Path?** Yes: $B_{\text{exp}} \cap B_0 = \{6\}$. Output the augmenting path

$$(6, f(6), f(f(6)), f(f(f(6)))) = (6, \mathsf{a}, 1, \mathsf{g}).$$

Of the edges on the augmenting path, edge a1 is part of M, while edges a6 and g1 are not. Therefore we improve our matching by removing a1 and adding a6 and g1. The result is shown in Figure 13.4b.

It is common to represent the exploration process by a forest, with the vertices arranged in levels. At the top level are the vertices in A_0; from there, the levels alternate between vertices added to B_{exp} in an **Explore** step and vertices added to A_{exp} in a **Match** step. The forest we get in this example is shown in Figure 13.4a: to make it easy to see the alternating paths, edges of the matching are highlighted.

Question:	The forest in Figure 13.4a looks like it has other augmenting paths, such as $(2, \mathsf{d}, 4, \mathsf{h})$. Could we have used this path instead?
Answer:	No: this path only looks like an augmenting path because we stopped before the next **Match** step, but in reality, vertex 2 is covered by M. If we had kept going, we would have found a longer augmenting path, which is in fact the path we'll find in the next round of the algorithm.

Question:	Vertices a, d, and f have more neighbors: why didn't we explore all of them?
Answer:	A mix of reasons. We did not explore 1 and 4 from a, even though they are adjacent to a, because they were already in B_{exp}. We did not explore 7 from d for a different reason: in the very same step, we already explored 7 from a. Each explored vertex x is explored from only one source $f(x)$, to make the augmenting path easy to find.

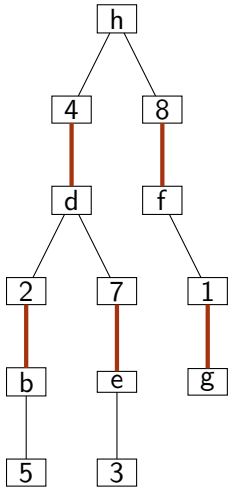
We have improved the matching, but we're still not done, so we continue with another round of the algorithm. The new matching M is $\{\mathsf{a6}, \mathsf{b2}, \mathsf{c3}, \mathsf{d4}, \mathsf{e7}, \mathsf{f8}, \mathsf{g1}\}$, so the uncovered vertices are given by $A_0 = \{\mathsf{h}\}$ and $B_0 = \{\mathsf{5}\}$. Now we:

1. **Explore** the vertices 4 and 8, adding them to B_{exp}, with $f(\mathsf{4}) = f(\mathsf{8}) = \mathsf{h}$.
2. **Check** the stopping conditions:
   - Is there a **Path?** No: $B_{\text{exp}} \cap B_0 = \varnothing$.
   - Is there a **Cover?** No: we added 2 new vertices to B_{exp}.
3. **Match** vertex 4 to d and vertex 8 to f. Add d and f to A_{exp}, with $f(\mathsf{d}) = \mathsf{4}$ and $f(\mathsf{f}) = \mathsf{8}$.
4. **Explore** the vertices 1, 2, and 7, adding them to B_{exp}, with $f(\mathsf{2}) = f(\mathsf{7}) = \mathsf{d}$ and $f(\mathsf{1}) = \mathsf{f}$.
5. **Check** the stopping conditions:
   - Is there a **Path?** No: $B_{\text{exp}} \cap B_0 = \varnothing$.
   - Is there a **Cover?** No: we added 3 new vertices to B_{exp}.
6. **Match** vertex 1 to g, vertex 2 to b, and vertex 7 to e. Add g, b, and e to A_{exp}, with $f(\mathsf{g}) = \mathsf{1}$, $f(\mathsf{b}) = \mathsf{2}$, and $f(\mathsf{e}) = \mathsf{7}$.
7. **Explore** the vertices 3 and 5, adding them to B_{exp}, with $f(\mathsf{3}) = \mathsf{e}$ and $f(\mathsf{5}) = \mathsf{b}$.
8. **Check** the stopping conditions:
   - Is there a **Path?** Yes: $B_{\text{exp}} \cap B_0 = \{\mathsf{5}\}$. Output the augmenting path

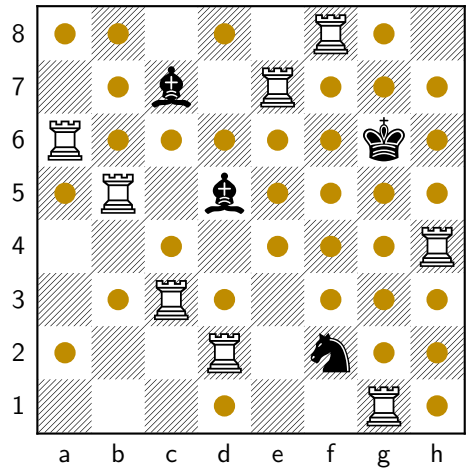
$$(\mathsf{5}, f(\mathsf{5}), f(f(\mathsf{5})), f(f(f(\mathsf{5}))), f(f(f(f(\mathsf{5})))), f(f(f(f(f(\mathsf{5})))))) = (\mathsf{5}, \mathsf{b}, \mathsf{2}, \mathsf{d}, \mathsf{4}, \mathsf{h}).$$

Of the edges on the augmenting path, edges b2 and d4 are part of M, while edges b5, d2, and h4 are not. Therefore we improve our matching by removing edges b2 and d4, then adding edges b5, d2, and h4 in their place. The result is shown in Figure 13.5b, with the exploration tree shown in Figure 13.5a.

We've found a perfect matching: we've placed 8 rooks on the chessboard, so we know that there can't be more. This means that we don't need to continue with another round of the augmenting path algorithm. Checkmate!



(a) The exploration tree



(b) Rooks on $\{\mathsf{a6}, \mathsf{b5}, \mathsf{c3}, \mathsf{d2}, \mathsf{e7}, \mathsf{f8}, \mathsf{g1}, \mathsf{h4}\}$

Figure 13.5: From 7 rooks to 8 rooks

Question:	If we did use the algorithm, what would happen?
Answer:	Since there are no uncovered vertices, we wouldn't explore from anywhere, the **Cover** stopping condition would hold in the very first **Check** step.

Question:	What vertex cover would we get?
Answer:	Since $A_{\text{exp}} = B_{\text{exp}} = \varnothing$, U would just be $A = \{\mathsf{a}, \mathsf{b}, \ldots, \mathsf{h}\}$.

13.6 Practice problems

1. Use the augmenting path algorithm to improve the matching in Figure 13.2a. Does this result in the perfect matching in Figure 13.2b, or a different one?

2. Let G be a bipartite graph with bipartition (A, B) and minimum degree $\delta(G)$.

   The sets A and B are guaranteed to be vertex covers; suppose U is a vertex cover that *does not* contain either A or B.

   a) Prove that in this case, $|U| \geq 2\delta(G)$.

   b) When does it follow that $\alpha'(G) \geq 2\delta(G)$, and why?

   c) In the case $|A| = |B| = 10$ and $\delta(G) = 3$, give an example of a bipartite graph in which this lower bound is true and $\alpha'(G) = 6$.

3. If G is not bipartite, then Proposition 13.1 still guarantees that $\alpha'(G) \leq \beta(G)$, but the two quantities may be different.

a) Find $\alpha'(G)$ and $\beta(G)$ when $G = C_{2k+1}$, a cycle graph with an odd number of vertices, in terms of k. Verify that they are, in fact, different.

b) Find a graph G which satisfies $\alpha'(G) = \beta(G)$, but is not bipartite.

4. Let G be a bipartite graph G with bipartition (A, B) which is r-regular: every vertex has degree r. Let $n = |A|$.

   a) Prove that $n = |B|$ as well.

   b) In terms of n and r, how many edges does G have?

   c) Prove that a set of k vertices in G can cover at most kr of the edges. What does this say about the size of a vertex cover?

   d) Use Kőnig's theorem to prove that G must have a perfect matching.

5. Prove if the augmenting path algorithm is applied to a maximum matching, then A_{exp} (the set of explored vertices on side A) is the set of all vertices on side A which are left uncovered by *some* maximum matching in the graph.

   (You must prove two things: that if $a \in A_{\text{exp}}$, then there is some maximum matching which does not cover a, and that if $a \in A - A_{\text{exp}}$, then all maximum matchings cover a.)

14 Hall's theorem

The purpose of this chapter

Applications of bipartite matchings outside of graph theory are usually all-or-nothing: either we find a perfect matching, or else we have made no progress. Additionally, the graphs we face are not arbitrary: they obey some mathematical laws defining their edges.

In such a scenario, Hall's theorem is usually the way to proceed. Hall's condition is a necessary and sufficient condition for the matching we want, and the abstract definition of our graph means it's often easy to verify. In this chapter, we will deduce Hall's theorem from Kőnig's theorem and then—because this is not enough to make a complete chapter—we will see some applications, ranging from recreational mathematics to serious combinatorics.

Hall's theorem is also often referred to as Hall's marriage theorem, and Hall's condition is then called the marriage condition. I'm not philosophically opposed to this (it can sometimes be handled poorly when teaching, but it can also be handled well), but I don't like the terminology, so I've left it out of this chapter.

14.1 Hall's theorem

Chapter 13 was devoted to proving Kőnig's theorem, which is the result you want to use for finding a maximum matching in a graph. In applications like the three-card magic trick, that is absolutely not the question we want to ask. We will only be satisfied with a perfect matching, because a magic trick that only works some of the time is no magic trick at all.

Of course, Theorem 13.2 can also tell us when a perfect matching exists, in terms of the number of vertices in a minimum vertex cover, but this feels like the wrong way of thinking. In a graph G with bipartition (A, B), where $|A| = |B| = n$, we want to distinguish between $\beta(G) = n$ and $\beta(G) \leq n - 1$ to determine whether a perfect matching exists, and this doesn't feel like a dramatic change. What's more, a vertex cover containing $n - 1$ vertices isn't a very intuitive explanation of *why* there is no perfect matching.

However, there are some situations in which we can tell a clear story about why a perfect matching fails to exist.

Question:	What if the graph contains an isolated vertex, x?
Answer:	Then there is no perfect matching, because there is no matching that covers x.

Question:	What if vertices x and y have degree 1 and share the same neighbor z?
Answer:	Then a matching can contain edge xz and cover x, or contain edge yz and cover y, but not both: so there is still no perfect matching.

Generalizing these concepts, we can describe a minimum requirement for a graph to have a perfect matching. First, we need a new definition.

Definition 14.1. *If G is a graph and $S \subseteq V(G)$ is a set of some of its vertices, then the* ***neighborhood*** *of S in G, denoted $N_G(S)$, is the set of all vertices adjacent to at least one vertex in S. If the graph G is clear from context, we simply write $N(S)$.*

For example, a vertex x is isolated if $N(\{x\}) = \varnothing$. Two leaf vertices x and y share their only neighbor z if $N(\{x, y\}) = \{z\}$. What do these situations have in common? A set of vertices has a neighborhood which is too small. This motivates the following condition:

Definition 14.2. *For a bipartite graph G with bipartition (A, B),* ***Hall's condition*** *is the condition that $|N(S)| \geq |S|$ for all subsets $S \subseteq A$.*

Question:	Suppose Hall's condition is violated for even one set $S \subseteq A$. Why is there no perfect matching?
Answer:	A perfect matching must pair each vertex of S with a different vertex in $N(S)$, which is impossible if there are not $\lvert S\rvert$ different vertices in $N(S)$ to use.

This explains why Hall's condition is necessary for a perfect matching to exist. Hall's theorem (proved by Philip Hall in 1935 [2]) states that the condition is also sufficient:

Theorem 14.1 (Hall's theorem)**.** *A bipartite graph G with the bipartition (A, B) has a matching that covers all vertices in A if and only if it satisfies Hall's condition.*

In particular, when $|A| = |B|$, G has a perfect matching if and only if it satisfies Hall's condition.

Question:	What if $\lvert A\rvert > \lvert B\rvert$?
Answer:	Then Hall's condition will not be satisfied if we take $S = A$, because $N(A) \subseteq B$ and therefore $\lvert N(A)\rvert < \lvert A\rvert$.

Question:	What if $\lvert A\rvert < \lvert B\rvert$?
Answer:	Then there is no perfect matching. In this case, Hall's theorem promises us a matching that covers every vertex in A, but leaves some vertices in B uncovered.

Applying Hall's theorem when $|A| < |B|$ can still be very useful if the vertices in A are different in type from the vertices in B. For example, in the magic trick, we really want a matching that covers every vertex $\{a, b, c\}$ corresponding to a set of 3 cards: then the matching tells my assistant what to do when handed such a set. It would be fine if the matching did not cover some ordered pairs (a, b): that would mean that my assistant will never hand me those two cards.

By proving Kőnig's theorem, we have done all the hard work of proving Hall's theorem, and can now deduce it as a corollary. (It would also have been possible to go the other way, and prove Kőnig's theorem as a corollary of Hall's theorem.)

Proof of Theorem 14.1. We already know that Hall's condition is necessary for a matching to cover A to exist: if Hall's condition is violated for some subset $S \subseteq A$, then there is not even a matching that covers every vertex in S. It remains to prove that Hall's condition is sufficient. We will do this by assuming that there is no matching that covers A, and finding a set S for which Hall's condition is violated.

Every edge in a matching has one endpoint in A and one endpoint in B. So if there is no matching that covers A, then there is no matching with $|A|$ edges: $\alpha'(G) < |A|$. By Theorem 13.2, $\beta(G) < |A|$: there is a vertex cover U such that $|U| < |A|$.

This set U is a small set of vertices with a lot of neighbors, since every edge of G has at least one endpoint in U. To find a set S for which Hall's condition is violated, we want the opposite: a large set of vertices with few neighbors. So we define S to be $A - U$: the set of all vertices in A that are not part of the vertex cover.

For all vertices $x \in S$, if y is adjacent to x, then U contains one endpoint of xy, but U does not contain x, so U must contain y. Therefore all of $N(S)$ is contained in U. In addition to the vertices of $N(S)$ (which are all on side B), U contains all of $A - S$ (on side A); therefore

$$|U| \geq |N(S)| + |A - S| = |N(S)| + |A| - |S|.$$

However, we also know $|U| < |A|$, so $|N(S)| + |A| - |S| < |A|$. This can be rearranged to get $|N(S)| < |S|$, proving that S violates Hall's condition. □

Question:	Suppose I give you a set S which violates Hall's condition. Can you use it to find a vertex cover with fewer than $\|A\|$ vertices?
Answer:	Yes: the set $(A - S) \cup N(S)$ is a vertex cover. For every edge xy, where $x \in A$ and $y \in B$, we have two cases: if $x \in S$, then y is in the vertex cover, and if $x \notin S$, then x is in the vertex cover.

14.2 Regular bipartite graphs

It is important to be clear about how Hall's theorem is meant to be used. It is not meant to be wielded like a hammer, going through the subsets of A one by one and searching for one that violates Hall's condition. Such an algorithm would take an exponentially long time! No: if you have a concrete graph in front of you, and it has no short mathematical description, use the augmenting path algorithm in Chapter 13 to find a maximum matching and a minimum vertex

cover. (There's one exception: if you can quickly spot a small set that violates Hall's condition, of course you should use that.)

Instead, we use Hall's theorem when dealing with a graph or a family of graphs in which the edges follow a regular mathematical pattern, and we can *prove* that Hall's condition is satisfied.

Our first example predates both Kőnig's theorem and Hall's theorem: it originally appeared in 1894, in a thesis by Ernst Steinitz about point-line configurations [10], though it was not stated in the language of graph theory.

Theorem 14.2. *For all $r \geq 1$, if G is an r-regular bipartite graph, then it has a perfect matching.*

I admit that I am so fond of this theorem that I've put two versions of it in an end-of-chapter exercise already, once in Chapter 8 and once in Chapter 13. But we have not had the opportunity to use Hall's theorem to prove it, so let's do that.

Proof of Theorem 14.2. Here's a starting observation that proves it's permissible to at least hope for a perfect matching. If G has a bipartition (A, B), then we can count the edges of G in two ways. First, we could sum the degrees of all vertices in A and get $r|A|$: this counts every edge once, because every edge has an endpoint in A. We could also sum the degrees of all vertices in B and get $r|B|$. The two methods must give the same answer, so $r|A| = r|B|$, and $|A| = |B|$.

To prove that a perfect matching exists, we show that Hall's condition holds for an arbitrary subset $S \subseteq A$. As in the previous paragraph, there are $r|S|$ edges with an endpoint in S. Similarly, there are $r|N(S)|$ edges with an endpoint in $N(S)$. By definition, every edge with an endpoint in S has the other endpoint in $N(S)$; in other words, the $r|N(S)|$ edges with an endpoint in $|N(S)|$ include among them all $r|S|$ edges with an endpoint in S. This gives the inequality $r|N(S)| \geq r|S|$, or $|N(S)| \geq |S|$. □

Theorem 14.2 is one of the few results in matching theory that is useful for multigraphs, not just simple graphs. You would not expect this to happen! After all, loops and parallel edges can never help us find a larger matching: a matching can never contain a loop (since the vertex incident on the loop would have degree 2 in the matching), nor can it use more than one edge between two vertices (since those edges share endpoints). In fact, bipartite graphs cannot contain loops in the first place, though they can have paralel edges: a loop can never have one endpoint in A and the other in B.

Question:	So why would it matter that Theorem 14.2 is true for multigraphs?
Answer:	It is possible that a bipartite multigraph is only r-regular due to the presence of parallel edges, and if we eliminate those, then checking Hall's condition becomes much harder.

To confirm for ourselves that Theorem 14.2 still does work for multigraphs, we could go through all our proofs and verify that nothing changes in the presence of parallel edges. (We would need to go through many proofs because of the dependencies between them.) However, there is a shortcut.

Let G be a bipartite r-regular multigraph, and let G' be the simple graph obtained by deleting all but one edge between every two adjacent vertices. We can quickly check that the two-paragraph proof of Theorem 14.2 still works for multigraphs, and therefore G satisfies Hall's condition. Therefore G' also satisfies Hall's condition: adjacent vertices in G are still adjacent in G', so neighborhoods don't change at all. Therefore G' has a perfect matching, which corresponds to a perfect matching in G.

It is possible to generalize the result of Theorem 14.2. Suppose that some vertices in A can have degree more than r, and some vertices in B can have degree less than r. Then $r|S|$ is a lower bound on the number of edges with an endpoint in S, while $r|N(S)|$ is an upper bound on the number of edges with an endpoint in $N(S)$, but this only extends the inequalities we have; we are still able to obtain $r|N(S)| \geq r|S|$ and verify Hall's condition.

Question:	Does this give us a perfect matching?
Answer:	No, because it's now possible that $\lvert A\rvert < \lvert B\rvert$.

However, we can still obtain the following corollary:

Corollary 14.3. *For all r, if G is a bipartite graph with bipartition (A, B) such that every vertex in A has degree at least r, while every vertex in B has degree at most r, then G has a matching that covers A.*

Armed with Hall's theorem and several of its variants, we are now ready to tackle some problems outside graph theory.

14.3 A three-card magic trick

Here is a mathematical magic trick I know. A version of it using the ordinary 52-card deck was invented by mathematician (and stage magician) Fitch Cheney [6], but I will present a simplified version with a deck of 8 cards, numbered 1 through 8.

You choose 3 cards from the deck, however you like, and give them to my assistant, while I look away or even temporarily leave the room. The assistant then hands me two of the cards, one at a time, and hands the third card back to you, for reference.

I have not seen the third card you're holding, at any point. Nevertheless, I can now perform some quick mental arithmetic and name the number on that card!

How is this possible? The answer doesn't involve any hidden communication: my assistant doesn't try to communicate the third card via eyebrow gestures and split-second timing. The answer involves only mathematics. Before describing a particular implementation of this trick, let's understand how it is even possible to make the trick work.

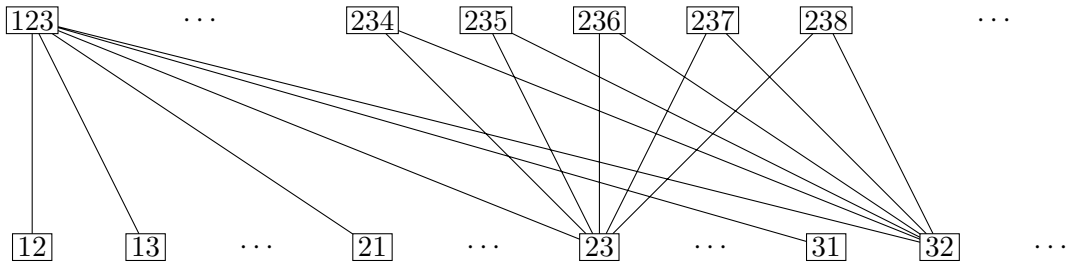


Figure 14.1: The bipartite graph for the three-card magic trick

Consider the following bipartite graph. On one side of the bipartition, the vertices are sets of three cards: $\{1,2,3\}$ through $\{6,7,8\}$. In the magic trick, you have chosen one of these vertices. On the other side, the vertices are *ordered pairs* of two cards: $(1,2)$, $(1,3)$, and so on through $(8,7)$. In the magic trick, this represents the information I get: the order in which the assistant hands me the cards makes the pair an ordered pair.

For the edges, join each set of three cards to the 6 ordered pairs that can be formed from it: these represent the 6 choices my assistant has, in deciding which two cards to give me and in which order. The graph has 112 vertices, so it is too big to draw in full, but Figure 14.1 shows a portion of it: enough to see all the neighbors of vertices $\{1,2,3\}$ on the first side and of $(2,3)$ and $(3,2)$ on the second side.

My assistant's strategy, whatever it might be, involves selecting one edge incident on each vertex on the first side; for example, selecting the edge between $\{1,2,3\}$ and $(1,2)$ means that if you choose the cards numbered 1, 2, and 3, my assistant will hand me card 1 and then card 2. Let M (for "magic") be the set of selected edges.

I presumably know M ahead of time, so if I am handed card 1 and then card 2, I should think about the edges of M incident on vertex $(1,2)$. If the edge between $\{1,2,3\}$ and $(1,2)$ is the only such edge in M, then I know that the third card must have been 3. If, however, there are multiple edges of M incident on $(1,2)$, then I have many theories, and cannot identify the third card. In other words, the magic trick only works if every vertex on the second side is also incident on at most one edge of M: if M is a matching!

Does such a matching exist? A first step is to check the number of vertices. There are $\binom{8}{3} = \frac{8\cdot7\cdot6}{3!} = 56$ vertices on the first side, and $8 \cdot 7 = 56$ vertices on the second side. So it's conceivable to have a matching M that covers all 56 vertices on the first side, as long as it also covers all 56 vertices on the second side: it must be a perfect matching. We haven't ruled out the possibility that the magic trick works.

Question:	Why do we divide by 3! in $\frac{8\cdot7\cdot6}{3!}$, but don't divide by anything in $8 \cdot 7$?
Answer:	In the first case, we're counting sets, which are unordered, so we divide by the 3! ways to order the elements. In the second case, we're counting ordered pairs: my assistant hands me a first card and a second card.

To see that the graph for the 8-card magic trick has a perfect matching, all we have to do is observe that it is 6-regular and apply Theorem 14.2.

Question:	Why is the graph 6-regular?
Answer:	For every pair of cards I see, there are 6 possibilities for the missing third card. For every three cards my assistant sees, there are 6 possible actions: 3 ways to choose the first card to pass me, multiplied by 2 ways to choose the second card.

We can generalize to a k-card magic trick: you choose k cards from a (potentially very large) deck, my assistant hands me $k-1$ of them in some order, and I must guess the last card.

Proposition 14.4. *The k-card magic trick can be performed if the deck contains at most $k! + k - 1$ cards.*

Proof. Suppose the deck contains n cards; construct the bipartite graph with bipartition (A, B) where A is the set of all unordered k-card hands, B is the set of all ordered sequences of $k-1$ cards, and there is an edge whenever a k-card hand contains all cards in the $(k-1)$-card sequence.

Then every vertex in A has degree $k \cdot (k-1)! = k!$: when my assistant takes a k-card hand and hands me $k-1$ cards in some order, there are k ways to choose which card I don't get to see, and $(k-1)!$ ways to sort the cards I do see. Every vertex in B has degree $n-k+1$: when I am handed $k-1$ cards, the number of k-card hands they could have come from is equal to the number of possibilities for the k^{th} card, which is $n-(k-1)$ because that card can be any of the cards except for the $k-1$ I see.

If we let $r = k!$, then every vertex in A has degree at least (actually, exactly) r. Provided $n-k+1 \leq r$, every vertex in B has degree at most r; this inequality is equivalent to $n \leq k!+k-1$. When this happens, by Corollary 14.3, the graph has a matching M that covers all vertices in A.

At least in principle, if my assistant and I agree on a matching M, then we have a strategy for the k-card trick. When handed k cards, my assistant finds the edge of M incident on that k-card hand, and looks at the other endpoint to see which $k-1$ cards to give me, and in which order. When I am handed that sequence of $k-1$ cards, I can also identify the edge of M containing that vertex. (If $n < k! + k - 1$, not all sequence of $k-1$ cards are covered by M, but every sequence my assistant actually hands is guaranteed to be covered!) The other endpoint of that edge tells me the original k-card set, so it tells me the hidden card. □

Question:	What's the problem with implementing this strategy?
Answer:	For an arbitrary matching, too much memorization is necessary! Essentially, my assistant and I have to memorize what to do in every possible situation.

In 2002, Michael Kleber [3] not only proved Proposition 14.4, but also described a strategy for the k-card trick that can be implemented in practice, and I will finish our discussion of magic tricks by describing it in the three-card case.

My assistant's job is to add up the numbers on the three cards, and find the remainder when the sum is divided by 3. The card my assistant hands back to you is the smallest of the three

cards if the remainder is 0, the middle card if the remainder is 1, and the largest card if the remainder is 2. My assistant hands me the other two cards in increasing order if the missing card is one of the three smallest cards I won't see, and in decreasing order otherwise.

Question:	What will my assistant do if you choose the cards $1, 3, 6$?
Answer:	$1 + 3 + 6 = 10$, which leaves a remainder of 1 when divided by 3, so my assistant will hand you the middle card: 3. Of the cards I won't see, the three smallest ones are $2, 3, 4$, of which 3 is one, so my assistant hands me 1, then 6.

My job is to label the unseen cards, in order from largest to smallest, as <2, <1, <0, >2, >1, and >0. I then guess the one where the inequality sign ($<$ or $>$) matches the relationship between the first and second card I am given, and the number (0, 1, or 2) matches the remainder when the sum of my two cards is divided by 3.

Question:	What will I do if I am handed the cards 1, then 6?
Answer:	Since $1 < 6$ and $1 + 6$ leaves a remainder of 1 when divided by 3, I want the card labeled <1: the second-smallest unseen card. So I guess card 3.

Though this strategy is workable, it is not the most elegant, and I wonder if—maybe just for the three-card trick—an even simpler procedure exists. Can you find one?

14.4 Generalized tic-tac-toe

The ordinary game of tic-tac-toe is played on a 3×3 grid. Players take turns placing their mark to claim an empty space in the grid: a $\times$ for the first player and a $\bigcirc$ for the second player. A player wins by claiming all three spaces in a horizontal, vertical, or diagonal line; if this does not happen when the board is filled, then the game is a draw.

The ordinary game of tic-tac-toe is not very interesting once you've learned how to play: every game between two skilled players ends in a draw. However, there are many ways to generalize it. The game gomoku is played on a 15×15 board, and the winning condition is to get 5 consecutive pieces in a row. Mathematicians have also tried playing tic-tac-toe in three (or more?) dimensions. A $3 \times 3 \times 3$ game is not very interesting, because the first player has an insurmountable advantage when claiming the center space, but $4 \times 4 \times 4$ tic-tac-toe is worth playing. You just have to learn to spot the winning lines (one example is shown in Figure 14.2).

We will consider tic-tac-toe in even further abstraction: there is a set of points $\mathcal{P}$, which the players take turns claiming, and a set of winning lines $\mathcal{L}$. Each element of $\mathcal{L}$ is a subset of $\mathcal{P}$, and a player that claims all the elements of a winning line wins.

Graph theory cannot solve all possible tic-tac-toe games at once, but an application of Hall's theorem can let us stop thinking about games that are too boring: either player can easily guarantee a draw by a strategy with very little interaction.

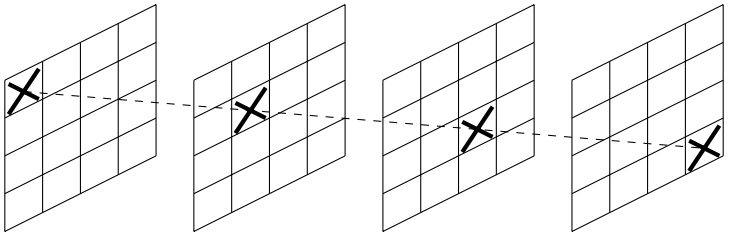


Figure 14.2: A winning line in $4 \times 4 \times 4$ tic-tac-toe

Proposition 14.5. *In a generalized tic-tac-toe game where every set of k winning lines contain at least $2k$ points in total, either player can force a draw.*

Proof. We will construct a somewhat unusual bipartite graph to represent the game. On one side of the bipartition, we will have $\mathcal{P}$: the set of points. On the other side of the bipartition, we will *not* put $\mathcal{L}$, but two disjoint sets called $\mathcal{L}^+$ and $\mathcal{L}^-$. For every winning line $\ell \in \mathcal{L}$, we put a "positive vertex" ℓ^+ into $\mathcal{L}^+$ and a "negative vertex" ℓ^- into $\mathcal{L}^-$. Both ℓ^+ and ℓ^- will be adjacent to the same set of points in $\mathcal{P}$: all the points contained in ℓ.

Next, we set out to check Hall's condition for a matching in this graph to cover $\mathcal{L}^+ \cup \mathcal{L}^-$. Let $S \subseteq \mathcal{L}^+ \cup \mathcal{L}^-$ be an arbitrary set. Then either $|S \cap \mathcal{L}^+|$ or $|S \cap \mathcal{L}^-|$ is at least $\frac{1}{2}|S|$: either at least half the vertices in S are positive, or at least half the vertices are negative. The two situations are symmetric, so we assume $|S \cap \mathcal{L}^+| \geq \frac{1}{2}|S|$.

If $k = |S \cap \mathcal{L}^+|$, then there are k winning lines corresponding to vertices in $S \cap \mathcal{L}^+$, and by the condition we've assumed, there are at least $2k$ points on these lines. All these points are in $N(S \cap \mathcal{L}^+)$, and therefore in particular they are in $N(S)$. Therefore $|N(S)| \geq 2k \geq |S|$, and Hall's condition holds.

Hall's theorem gives us a matching in our unusually constructed graph, but what do we do with this matching? Well, for every line ℓ, the matching gives us two points on ℓ: the point matched to ℓ^+ and the point matched to ℓ^-. In other words, from every winning line we've selected two points, such that each point has been selected from at most one of the winning lines through it.

Using these selected points, you can force a draw whether you are the first player or the second. Suppose your opponent has played on one of the points selected from winning line ℓ. Then respond by claiming the other point selected from ℓ, if you have not claimed it already. In all other cases—if your opponent's move is not the point selected from any line, or if you've already claimed the other point, or if it's the first move of the game—play arbitrarily. (A strategy of this type is called a **pairing strategy** in game theory.)

Question:	What if your opponent plays on that point with the goal of winning on some line other than ℓ?
Answer:	You don't care, because that other line has two selected points of its own.

The game will end in a draw if you use the pairing strategy, because your opponent can never claim all the points on any line. To do so, eventually your opponent would need to claim one of the selected points from that line, but then you will just claim the other point. □

The condition of Proposition 14.5 seems hard to check, but for many tic-tac-toe games, it turns out k winning lines will contain far more than $2k$ in most cases, leaving only a few cases to check. For example, consider a 5×5 board. A single line is guaranteed to have 5 points; a second line is guaranteed to add 4 new ones; a third line is guaranteed to add 3 new ones, for a total of 12.

Question:	How can we argue rigorously that 3 lines contain at least 12 points?
Answer:	They have $5 + 5 + 5 = 15$ points if we double-count the intersections, and 3 lines have at most 3 intersection points.

Since 12 points is enough for up to 6 lines, we can skip ahead to considering a set of 7 lines. These cover most of the board, and with a little bit more thinking, we can conclude that the worst case is a set of all $k = 12$ lines, which cover all 25 points. Therefore Proposition 14.5 applies to tic-tac-toe on a 5×5 board.

14.5 Incomparable sets

Our setting in this section will be the world of subsets of an n-element sets; we might as well assume they are subsets of $\{1, 2, \ldots, n\}$, because the exact elements won't matter.

Two subsets X and Y are called **comparable** if $X \subseteq Y$ or $Y \subseteq X$, and **incomparable** otherwise. For example, the set $\{2, 3\}$ is comparable to (and a subset of) the set $\{2, 3, 4\}$; it is incomparable to $\{1, 3, 4\}$. You can imagine that if the elements represent objects you want to own, then $\{2, 3\}$ is clearly not as good as $\{2, 3, 4\}$, but it is not certain which of $\{2, 3\}$ or $\{1, 3, 4\}$ is better. For example, what if elements 1, 3, and 4 are different kinds of cookies, while element 2 is an expensive car?

Suppose we want to pick a family[1] of sets in which no two sets are comparable. An example is the family $\big\{\{1, 2, 3\}, \{2, 4\}, \{1, 3, 4\}\big\}$.

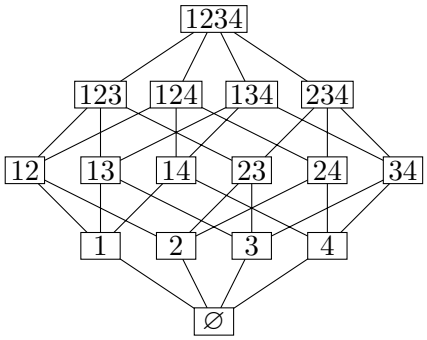
Question:	What is largest family of subsets of $\{1, 2, 3, 4\}$ in which no two are comparable?
Answer:	$\big\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\big\}$, the family of all 2-element subsets.

In general, if we take the family of all k-element subsets, for any k, then any two subsets in the family will be incomparable. There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ sets in this family. This is maximized when $k = n/2$: for example, when $n = 4$ we have

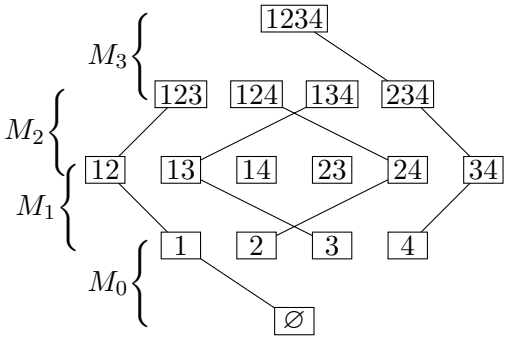
$$\binom{4}{0} = 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = 6, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1.$$

So we want the family of all $(n/2)$-element subsets. We will call this the **middle layer** family.

[1] A "family" of sets is just a set of sets, but we give it a different word to make it easier to distinguish it from its elements.



(a) Q_4, interpreting vertices as subsets



(b) $P = M_0 \cup M_1 \cup M_2 \cup M_3$

Figure 14.3: Diagrams accompanying the proof of Theorem 14.6

Question:	What if n is odd?
Answer:	When n is odd, there are two equally good middle layers: the family of $\frac{n-1}{2}$-element subsets, and the family of $\frac{n+1}{2}$-element subsets. (For example, if $n = 5$, then $\binom{5}{2} = \binom{5}{3} = 10$.

By convention, let's round $n/2$ down if n is odd. The notation for rounding down is $\lfloor n/2 \rfloor$, so the formula $\binom{n}{\lfloor n/2 \rfloor}$ tells us the size of the middle layer family.

In 1928, Emanuel Sperner proved [9] that this is the best we can do, for any n. He did not use Hall's theorem to do this, but we will.

Theorem 14.6. *If $\mathcal{F}$ is a family of subsets of $\{1, 2, \ldots, n\}$ such that no two different sets $X, Y \in \mathcal{F}$ are comparable, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

Proof. In Chapter 4, I mentioned that the hypercube graph Q_n can be useful when reasoning about set families, and that's what we will do here. We defined the vertex set of Q_n to be the set of n-bit strings: sequences $b_1 b_2 \ldots b_n$ where each b_i is either 0 or 1. To relate this to set families, a vertex $b_1 b_2 \ldots b_n$ can be identified with the subset of $\{1, 2, \ldots, n\}$ containing an element i whenever $b_i = 1$. The edges, which join n-bit strings that differ in only one position, now tell us which two sets differ only by the presence or absence of one element. For the rest of the proof, I will switch to using set-related terminology. Figure 14.3a shows, for example, Q_4 with the vertices interpreted as subsets (and $\{x, y, z\}$ written as xyz to simplify the diagram).

We proved in Proposition 12.2 that Q_n as a whole is bipartite, but a perfect matching in Q_n will not help us. Instead, we will write Q_n as a union of bipartite graphs:

$$Q_n = G_{n,0} \cup G_{n,1} \cup \cdots \cup G_{n,n-1}$$

where $G_{n,k}$ is the subgraph of Q_n induced by the sets of size k or $k + 1$. In other words:

- $G_{n,k}$ has the bipartition (A, B) where A is the set of k-element subsets of $\{1, 2, \ldots, n\}$, and B is the set of $(k + 1)$-element subsets;
- An edge xy with $x \in A$ and $y \in B$ exists whenever we can add one more element to x to make y. (It will be important later that when this happens, x is a subset of y.)

In $G_{n,k}$, every vertex $x \in A$ has $n-k$ neighbors in B: there are $n-k$ elements of $\{1, 2, \ldots, n\}$ not already in x which we can add. Every vertex $y \in B$ has $k+1$ neighbors in A: it has $k+1$ elements which we can remove. Provided $n-k \geq k+1$, or $k \leq \frac{n-1}{2}$, Corollary 14.3 applies, giving us a matching in $G_{n,k}$ that covers A.

Question:	What kind of matching can we get if $k \geq \frac{n-1}{2}$?
Answer:	In this case, Corollary 14.3 would apply if we switched the roles of A and B, so there is a matching that covers B.

In either case, call this matching M_k. As a general rule, M_k covers the side of $G_{n,k}$ with fewer elements, which is the side further from the middle layer(s).

Now the magic happens! Consider the union $M_0 \cup M_1 \cup M_2 \cup \cdots \cup M_{n-1}$, and call it P, because as we're about to see, it consists of many disjoint paths. Figure 14.3b shows one possible union P in the case $n = 4$.

Consider a vertex of Q_n corresponding to a subset of size k. This vertex has degree 1 or 2 in P: if $k \leq \frac{n-1}{2}$, it is incident on one edge in M_k and maybe also one edge in M_{k-1}, while if $k \geq \frac{n-1}{2}$, then it is guaranteed one edge in M_{k-1} and possibly also an edge in M_k. We can follow these edges "up" the graph (increasing the number of elements) and "down" the graph (decreasing the number of elements) until we hit a dead end—a vertex of degree 1. The result is a connected component isomorphic to a path graph.

How many connected components are there? Well, from each vertex, we can always follow edges of P to get closer to the middle layer(s): those are the guaranteed edges. Therefore each connected component of P contains one vertex in the middle layer (or in each middle layer, for odd n). There are $\binom{n}{\lfloor n/2 \rfloor}$ vertices in the middle layer, and therefore there are $\binom{n}{\lfloor n/2 \rfloor}$ components.

So far, we have only looked at the structure of subsets of $\{1, 2, \ldots, n\}$, but now we are ready to consider the set family $\mathcal{F}$ that Theorem 14.6 is ostensibly all about. You see, $\mathcal{F}$ can never contain two sets in the same connected component of P. If it did, then we could start at the set with fewer elements, and follow it up the path to get to the one with more elements. At each step, we're adding an element, so the first set will be a subset of the second—but we've assumed that $\mathcal{F}$ does not contain two such sets.

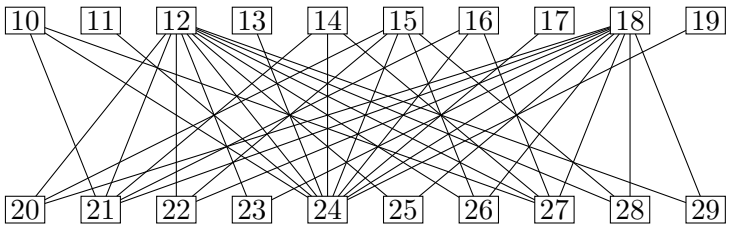
Therefore $\mathcal{F}$ contains at most one set from each connected component of P; since there are $\binom{n}{\lfloor n/2 \rfloor}$, we know that $\mathcal{F}$ contains at most $\binom{n}{\lfloor n/2 \rfloor}$ elements. □

14.6 Practice problems

1. Is it possible to circle a letter in each of the words below so that all 9 circled letters are different? Either find a way to do it, or prove that it's impossible using Hall's condition.

AREA	APART	ERRATA
GRAPH	PAPER	RETREAT
THEORY	TREE	YOGA

2. The multiples-of-six graph from Lecture 13 is shown again below:



   Prove that it does not have a perfect matching, but this time, using Hall's theorem.

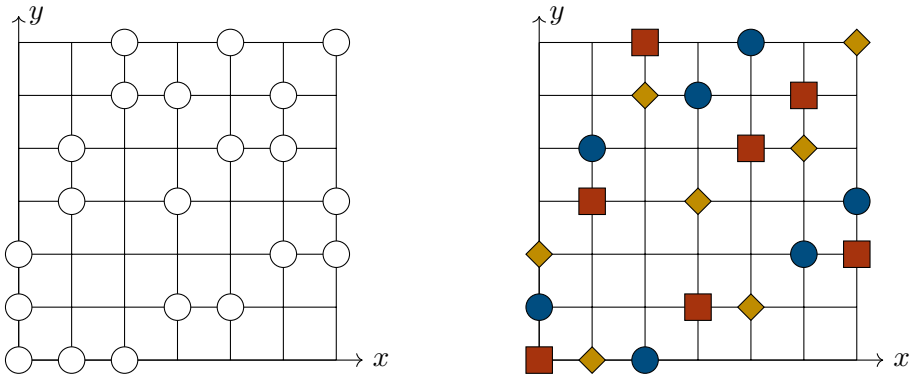
3. Let G be a bipartite graph with n vertices on each side of the bipartition whose minimum degree $\delta(G)$ is greater than $n/2$.

   Prove that G has a perfect matching.

4. Prove that when $n > k! + k - 1$, it is impossible to perform the k-card magic trick with an n-card deck and guarantee that I guess the k^{th} card. *(This is not a problem about graph theory; it is a matter of counting.)*

5. Find a pairing strategy for tic-tac-toe on a 5×5 board, where the goal is to win by claiming all 5 spaces along a horizontal, vertical, or diagonal line.

6. Suppose that we mark several points at integer coordinates in the xy-plane in such a way that for every marked point (a, b), the lines $x = a$ and $y = b$ each contain two other marked points. This can be done by making a 3×3 grid, or in other, more complicated ways, such as in the first diagram below.



   Prove that it is guaranteed to be possible to give the marked points 3 different colors, as in the second diagram above, so that no horizontal or vertical line passes through two marked points of the same color.

7. The following problem appeared on the 2012 William Lowell Putnam Mathematics competition [8]:

   A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

8. An $n \times n$ **Latin square** is an $n \times n$ grid filled with the integers 1 through n in such a way that every row and every column contains each integer exactly once. For example, the first 5×5 grid below is a Latin square.

1	2	3	4	5
3	1	4	5	2
5	3	1	2	4
2	4	5	1	3
4	5	2	3	1

1	2	3	4	5
5	3	4	1	2

Suppose that, as in the second grid above, the first r rows in the $n \times n$ grid are filled with integers 1 through n in a way that does not cause any contradictions: each of the r rows use each integer exactly once, and no column contains any duplicates. Prove that we can fill in the last $n - r$ rows to get a Latin square.

15 Matchings in general graphs

The purpose of this chapter

In the previous three chapters, we discussed matchings in bipartite graphs; here, we are going to take another chapter to consider matchings in general graphs. This division is not because the generalization is easy, but because it is hard. We will have to do quite a bit of work just to prove Tutte's theorem: the equivalent of Hall's theorem in cases where the graph is not necessarily bipartite.

There is a lot more that we're going to leave undone. We proved Kőnig's theorem which deals with the size of a maximum matching; we will not prove the Tutte–Berge formula, which generalizes Tutte's theorem to a Kőnig-type result. We used an algorithm to find maximum matchings in bipartite graphs. There is a more general algorithm, known as the **blossom algorithm**, which can handle non-bipartite graphs, but we will not discuss it; it is considerably more complicated. If you would like to learn more about matchings, I recommend Lovász and Plummer's book *Matching Theory* [7].

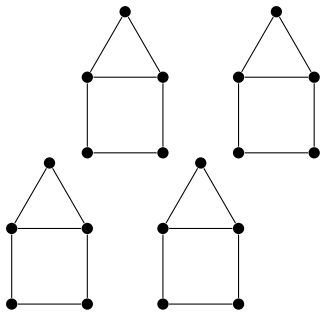
Compared to this, the second half of the chapter on 1-factorizations is less intense, and gives some graph-theoretical solutions to problems that come up in real life. A more relaxed path through this chapter is to learn about Tutte sets and the statement of Theorem 15.1, then dive into 1-factorizations and the geometric proof of Theorem 15.3.

There are a lot of other problems about factorizations I could have included instead of the section on increasing walks. In the end, though, I felt that many people could write a chapter about decomposing K_n into triangles or spanning cycles, but if I did not mention this beautiful problem in my textbook, it is not likely that someone else would write about it in theirs.

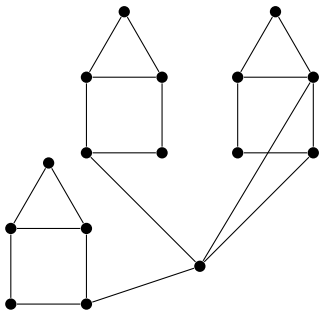
15.1 Tutte sets

In the case of bipartite graphs, we have a complete list of all the reasons that a perfect matching might fail to exist: the violations of Hall's condition. We can summarize these violations as problems of *insufficiency*: some vertices do not have enough neighbors for us to get a perfect matching.

If our graphs don't have to be bipartite, a second reason that we might not have a perfect matching appears, and that is *parity*. The complete graph K_{99}, or indeed K_{2n+1} for any n, does not have a perfect matching, even though it has all the edges we could wish for, simply because the number of vertices is odd. Each edge in a matching covers 2 vertices, and so the number of vertices covered by a matching is always even.



(a) Four houses



(b) Three houses and one more vertex

Figure 15.1: Obstacles to the existence of a perfect matching

Question:	Why didn't we need to worry about parity when proving theorems about bipartite graphs?
Answer:	In a graph with bipartition (A, B), we already know that a perfect matching only exists if $\|A\| = \|B\|$. When this happens, the total number of vertices is guaranteed to be even.

If we only had to worry about whether the total number of vertices is odd or even, that wouldn't be so bad. But there are examples showing that our life can be even worse:

Question:	In Figure 15.1a, the total number of vertices is even, and yet there's still a parity issue stopping us from having a perfect matching. How?
Answer:	Each house is a connected component with an odd number of vertices, so it has no perfect matching, and different houses cannot help each other.

We will use to this idea later throughout this chapter, and so we will say that a component with an odd number of vertices is an **odd component**, for short. However, odd components all by themselves are not the only problem.

Question:	In Figure 15.1b, there is a parity issue even though the graph is connected. How?
Answer:	Each of the three houses has no perfect matching on its own, and the extra vertex can be matched to only one of the houses.

We can imagine that the three houses in Figure 15.1b are on fire, and the vertex in place of the fourth house is a superhero that can put the fires out. However, the superhero can only be in one place, and cannot put out all the fires. This problem is a combination of parity and insufficiency: we have three parity problems in the graph, but only one vertex that can help us fix them.

The English–Canadian mathematician William Thomas Tutte generalized these problems. We can think of Hall's theorem as saying that if a bipartite graph does not have a perfect matching,

then we can summarize why, by giving an example where Hall's condition fails. Similarly, Tutte's 1947 theorem [11] says that if an arbitrary graph does not have a perfect, then we can summarize why, by showing a particular kind of obstacle.

We say that a set of vertices U in a graph G is a **Tutte set** if the graph $G - U$ has more than $|U|$ odd components.

Question:	Is there a Tutte set in Figure 15.1b?
Answer:	Yes: the "superhero" vertex in the bottom right.

Question:	Is there a Tutte set in Figure 15.1a?
Answer:	Yes, and the simplest is the empty set! (This is allowed, and sometimes it is the only option.)

Theorem 15.1. *Every graph G has a perfect matching if and only if it has no Tutte set.*

As with Hall's theorem, one direction of Theorem 15.1 is much shorter than the other, and the proof of that direction is what motivated our definitions: we defined a Tutte set specifically because we had an argument in mind for why a graph with a Tutte set could not have a perfect matching.

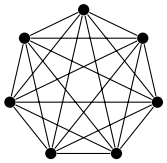
Proof of the "only if" direction of Theorem 15.1. Suppose that U is a Tutte set in a graph G, so that $G - U$ has $k > |U|$ odd components. Let M be a maximum matching in G, and let M' be the largest subset of M that is also a matching in $G - U$. Then M' must leave at least k vertices of $G - U$ uncovered: at least one from each odd component. However, M cannot do much better! An edge of M is missing from M' only if it has an endpoint in U, and there are at most $|U|$ such edges; these can cover at most $|U|$ of the vertices of $G - U$ that M' left uncovered. Since $k > |U|$, we know that there are still some vertices of $G - U$ that are uncovered by M; therefore M is not a perfect matching. □

To prove the other direction of Theorem 15.1, we will not use Tutte's original argument, which relied on linear algebra. Instead, we will see an argument from Lovász and Plummer's *Matching Theory*. We will arrive at it indirectly, by first describing a special class of graphs called "saturated graphs" for which the proof will be easier.

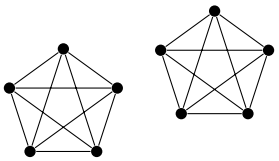
15.2 Saturated graphs

Call a graph G **saturated**[1] if G has no perfect matching, but if e is any edge not already present in G, then adding e to G would create a perfect matching. (This includes the case of

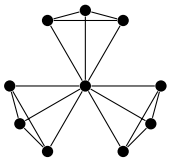
[1] The word "saturated" is commonly used in graph theory for this type of definition, but it usually needs to be qualified with some other adjectives to say what kind of problem G is saturated for. In this book, we will not need this term outside this chapter, so I will just say "saturated" for brevity, even though it's not as precise.



(a) K_7



(b) Two copies of K_5



(c) A nonempty Tutte set

Figure 15.2: Several examples of saturated graphs

odd complete graphs K_{2n+1}: they are saturated because they do not have a perfect matching, and there is no edge e that can be added.)

Starting from any graph that does not have a perfect matching, we can arrive at a saturated graph by adding edges, one at a time, for as long as this is possible without creating a perfect matching.

Question:	How could we do that in Figure 15.1b?
Answer:	We could add edges to each house to make it a copy of K_5, and then add edges from the extra vertex to every vertex of each house.

Figure 15.2 shows several more examples of saturated graphs. The idea of Tutte sets helps us come up with more: if we have a Tutte set, we can add any edges that don't change its structure.

Question:	If we start with the four houses in Figure 15.1a, and turn each house into a copy of K_5, is the result saturated?
Answer:	No: if we add an edge between two houses, that increases the size of the matching, but it's still not a perfect matching. We can go all the way up to a graph with a copy of K_5 and a copy of K_{15} before it's saturated.

Using Theorem 15.1, we can describe what saturated graphs look like without too much trouble. (Proposition 15.2 does not describe saturated graphs completely, but it will be enough for us.)

Proposition 15.2. *If a graph G is saturated, then for some set of vertices U, G contains every edge with at least one endpoint in U, and every component of $G - U$ is a complete graph.*

Proof of Proposition 15.2 using Theorem 15.1. If a graph G is saturated, then it has no perfect matching, so by Theorem 15.1, it must have a Tutte set U. If an edge e can be added to G that would not change U's status as a Tutte set, then the resulting graph $G + e$ still wouldn't have a perfect matching (again, by Theorem 15.1), which would violate the definition of a saturated graph. Therefore every such edge must already exist in G. Which edges are these?

Well, every edge with at least one endpoint in U is such an edge, because adding it wouldn't affect $G - U$, so U would still be a Tutte set. Also, every edge within a component of $G - U$ is such an edge, because adding it wouldn't affect the number of vertices in that component of $G - U$.

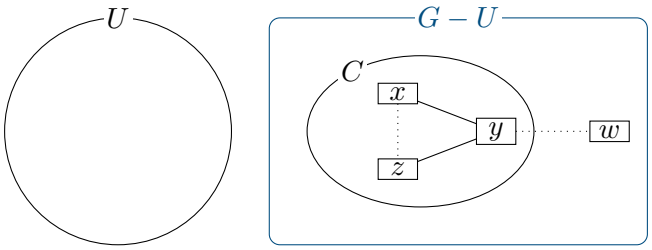


Figure 15.3: Vertices x, y, z, w in the proof of Proposition 15.2; edges xz and yw are *not* present

Therefore every component of $G - U$ is complete, and all edges with an endpoint in U are present in G. □

I made a point to say that this proof used Theorem 15.1, because it means that the proof is not good enough for our purposes. What are those purposes? We are going to reverse the logic, using Proposition 15.2 to prove Theorem 15.1!

15.3 Proof of Tutte's theorem

Let's begin with a fresh proof of Proposition 15.2 which we can use to deduce Theorem 15.1 without making the argument circular.

Proof of Proposition 15.2 without using Theorem 15.1. Start by taking U to be the set of all vertices that are adjacent to every other vertex.

Suppose that not every component of $G - U$ is a complete graph. Let C be an "incomplete" component of $G - U$; this means that we can find vertices x, y, and z such that x is adjacent to y, and y is adjacent to z, but x is not directly adjacent to z. Also, because $x, y, z \notin U$, y must not be adjacent to every vertex; there must be a fourth vertex w not adjacent to y. This is illustrated in Figure 15.3.

We have two ways to make G bigger: we could add edge xz, or we could add edge wy. Because G is saturated, each of these bigger graphs must have a perfect matching; because G has no perfect matching, those two matchings must each use the added edge. So if we remove the added edge, we see that G has two matchings that are *nearly* perfect: a matching M_{xz} that covers all vertices except x and z, and a matching M_{wy} that covers all vertices except w and y.

In Chapter 13, to compare two matchings, we looked at their symmetric difference, and we will do the same here. We know in general that $M_{xz} \oplus M_{wy}$ consists of cycles and alternating paths, but in this case, we know more.

Question:	What are the endpoints of the alternating paths in $M_{xz} \oplus M_{wy}$?
Answer:	They are, in some order, x, y, z, and w. All other vertices have degree 1 in both M_{xz} and M_{wy}, so they have degree 2 or 0 in $M_{xz} \oplus M_{wy}$.

So $M_{xz} \oplus M_{wy}$ contains an M_{wy}-alternating path that starts at vertex w and ends at x, y, or z. All of these are almost as good for us:

- If it ends at y, then it's an M_{wy}-augmenting path, since it starts and ends at an uncovered vertex.
- If it ends at x or z, then we can follow up with edge xy or zy to go to y; again, we get an M_{wy}-augmenting path.

Question:	If we follow up a $w - x$ alternating path by going from x to y, why is that still alternating?
Answer:	We must have arrived to x by an edge of M_{wy}, because M_{xz} leaves x uncovered. Meanwhile, xy is not an edge of M_{wy}, because M_{wy} leaves y uncovered.

By Lemma 13.3, we can use the M_{wy}-augmenting path to improve M_{wy} to a perfect matching—in the graph G, which we assumed was saturated, and didn't have a perfect matching! This is the contradiction that finishes the proof. □

Let me explain the reasons behind what's happened so far. It's common that when proving a theorem about graphs that don't have a property (such as a perfect matching), it is enough to prove the theorem about graphs that don't have the property, but have as many edges as possible: in our case, about saturated graphs. We're about to see in a moment that this is true here: proving Theorem 15.1 will be much easier for saturated graphs than for graphs with no special properties.

So we want to learn about saturated graphs. Since we suspect that Theorem 15.1 is true even before we have a proof, we started by using it to understand what saturated graphs *ought* to look like. Then, we went back and proved the same result in legitimate ways, so that it's no longer circular reasoning to use it to prove Theorem 15.1.

Now let's see if our efforts have paid off!

Proof of the "if" direction of Theorem 15.1. Let G be a graph that does not have a perfect matching.

Question:	How can we finish the proof if G has an odd number of vertices?
Answer:	In this case, $U = \varnothing$ is a Tutte set in G, so Theorem 15.1 holds.

So we assume that G has an even number of vertices. Then the reason that G doesn't have a perfect matching is simply because it's missing some edges.

Greedily add edges to G, one by one, for as long as this does not create a perfect matching, until we cannot add edges any longer. The result is a saturated graph H that contains G as a spanning subgraph.

By Proposition 15.2 applied to H, there is some set of vertices U such that H contains every edge with at least one endpoint in U, and every component of $H - U$ is a complete graph. We will first prove that U is a Tutte set in H, and then that it is a Tutte set in G.

Let M be a maximum matching in $H - U$. Here, all components are complete graphs, so the only reason some of the might not be entirely covered by M is because they're odd. What's more, if there are $|U|$ or fewer odd components, then we can extend M to a perfect matching of H: match vertices in U to vertices outside U not covered by M, and if any vertices in U are left, match them to each other. So there must be more than $|U|$ odd components in $H - U$, which exactly means that U is a Tutte set in H.

Question:	Where did we need the assumption that G and H have an even number of vertices?
Answer:	Otherwise, "if any vertices in U are left, match them to each other" might not work: we might end with one vertex uncovered.

Since G is a subgraph of H, every component of $H-U$ breaks down into one or more components of $G - U$. However, an odd number cannot be the sum of many even numbers; therefore every odd component of $H - U$ must contain at least one odd component of $G - U$. We conclude that $G - U$ has at least $|U|$ odd components, which means that U is also a Tutte set in G. □

15.4 1-Factorizations

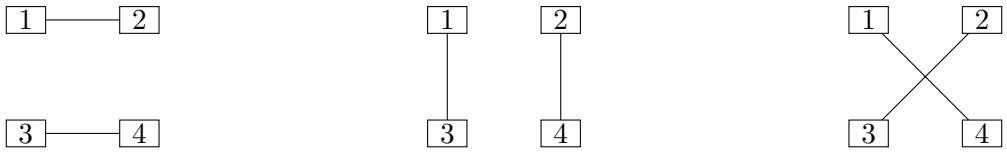
A practical application of matchings in graphs that we have yet to consider is tournament design. (We will only explore the basics of the overlap between this field and graph theory.)

Suppose that we would like to organize a round-robin chess tournament[2] between n players. A single chess game can take a while, so we want to schedule as many games at the same time as possible. When n is even, we can always begin by dividing n people into $n/2$ arbitrary pairs and scheduling a game between each pair. This is a perfect matching in K_n, which is not a very difficult matching problem.

However, this only tells us what to do in the first round; future rounds are more difficult, because we don't want to repeat any games. If the first round is a perfect matching M_1 in K_n, then we would like the second round to be a perfect matching M_2 in $K_n - M_1$, the third round to be a perfect matching in $K_n - M_1 - M_2$, and so on. These matching problems can easily become impossible if we schedule rounds of the tournament with no foresight.

For example, with 6 players, we hope to finish in 5 rounds, because each player has 5 opponents to face. However, if our first three matchings are poorly chosen, then they might leave us with a graph that has no perfect matching of its own. We would have to ask two people to sit out of the fourth round, and then we would have to schedule 6 rounds total.

[2] "Round-robin", in case you're unfamiliar with the terminology, means that every participant plays a game against every other participant.



(a) The 1-factor $(x_1x_2)(x_3x_4)$ (b) The 1-factor $(x_1x_3)(x_2x_4)$ (c) The 1-factor $(x_1x_4)(x_2x_3)$

Figure 15.4: A 1-factorization of K_4: three disjoint 1-factors whose product is $x_1^3x_2^3x_3^3x_4^3$

Question:	Can this problem really occur? How do we run a 6-player tournament badly?
Answer:	Taking $V(K_6)$ to be $\{1,2,3,4,5,6\}$, we might mistakenly begin with three matchings that all match even numbers to odd numbers: for example, $\{12,34,56\}$, then $\{14,25,36\}$, then $\{16,23,45\}$. Now, the remaining graph consists of two copies of K_3 on vertices $\{1,3,5\}$ and $\{2,4,6\}$, with no perfect matching.

Instead of solving this problem one matching at a time, we will have to solve it all at once: we want to find a **decomposition** of the edges of G into disjoint perfect matchings. There is a special term for this kind of decomposition.

Definition 15.1. *A* **1-*factorization*** *of a graph G is a decomposition of G into perfect matchings: a partition*

$$E(G) = M_1 \cup M_2 \cup \cdots \cup M_k$$

where each edge set M_i is a perfect matching, and each edge of G appears in exactly one M_i.

The term "1-factorization" goes back to the early days of graph theory, where the idea of a graph was more algebraic: an edge xy was really thought of as the product of two variables x and y, and a graph was just the product of such edges. For example, a cycle with vertices x, y, z might be the expression $(xy)(xz)(yz)$. Some other modern terms have their origin in those days. For example, if you simplify the product that we use to represent our cycle, you get $x^2y^2z^2$; the degree of each variable (in the algebraic sense) is exactly the degree of the corresponding vertex (in the graph-theoretic sense)!

Viewed from this point of view, a 1-factorization really is a factorization of the graph: a factorization into factors called **1-factors** in which every variable has degree 1. For example, the graph K_4, viewed as the product $(x_1x_2)(x_1x_3)(x_1x_4)(x_2x_3)(x_2x_4)(x_3x_4)$, has the 1-factorization

$$\underbrace{(x_1x_2)(x_3x_4)}_{\text{1-factor}} \cdot \underbrace{(x_1x_3)(x_2x_4)}_{\text{1-factor}} \cdot \underbrace{(x_1x_4)(x_2x_3)}_{\text{1-factor}}.$$

Figure 15.4 shows this factorization in a more modern way.

Of course, what's important is finding the 1-factorization, not representing it.

Theorem 15.3. *The complete graph K_n has a 1-factorization whenever n is even.*

Proof. It is much easier to draw a diagram of the 1-factorization than it is to give a diagram-free proof. Begin with a diagram of K_n that places $n-1$ of the vertices at evenly spaced points around a circle, and the last vertex in the middle of the circle. For the i^{th} matching M_i, take the edge between the middle vertex to the i^{th} vertex around the circle, as well as all edges perpendicular to this edge in the diagram.

(Figure 15.5 shows an example of this construction in the case $n = 8$.)

In a way, the diagram also leads to a geometric proof. Name the $n-1$ matchings after the $n-1$ radial edges. For each edge xy between two outer vertices, construct the diameter perpendicular to xy, and one half of that diameter will be an edge from the middle vertex. This tells us which matching contains edge xy; in particular, it tells us that xy is in exactly one matching. To see that it's a matching, we first check that xy does not share a vertex with the radial edge perpendicular to it (the lines intersect at the midpoint of edge xy, not at a vertex). In all other cases, two edges xy and $x'y'$ in the same matching are parallel: they do not share a vertex because, as lines, they do not intersect.

For a diagram-free proof, we rely on modular arithmetic instead. Number the vertices 0 through $n-1$. For each $i = 0, 1, \ldots, n-2$, define the matching M_i to contain the edge $\{i, n-1\}$ as well as all the edges $\{i-k \bmod n-1, i+k \bmod n-1\}$ for $k = 1, \ldots, n/2-1$. Since the $n-1$ values

$$i-(n/2-1), i-(n/2-2), \ldots, i-1, i, i+1, \ldots, i+(n/2-1)$$

are distinct modulo $n-1$, no vertices are repeated, and therefore M_i really is a matching.

Next, we show that no edge is contained in multiple matchings. This is true for edges incident on vertex $n-1$, since each matching is defined to contain a different one of these edges. Otherwise, suppose that an edge $\{x, y\}$ is in both M_i and M_j. Since it is in M_i, then $\{x, y\} = \{i-k \bmod n-1, i+k \bmod n-1\}$ for some k, so $x+y \equiv (i-k)+(i+k) = 2i \pmod{n-1}$. Similarly, since it is in M_j, then $x+y = 2j \pmod{n-1}$. But n is even and $0 \le i, j \le n-2$, so $2i \equiv 2j \pmod{n-1}$ can only occur if $i = j$.

Question:	What goes wrong at this step if n is odd?
Answer:	Then $2i \equiv 2j \pmod{n-1}$ really is possible for two values i and j. For example, if $n = 7$, we'd be working modulo 6, and $2 \cdot 1 \equiv 2 \cdot 4 \pmod 6$.

From the previous paragraph, it also follows that every edge is contained in some matching. If there are $n-1$ matchings, and each contains $n/2$ edges, then together they contain $\frac{n(n-1)}{2}$ edges total, and we've shown that there's no overlap. But there are only $\frac{n(n-1)}{2}$ edges in K_n, so we've included them all. □

In my opinion, the geometric proof is the "real" reason that Theorem 15.3 is true, but I have included the number-theoretic proof for completeness. It requires some background in number theory to follow, but in reality, working modulo $n-1$ is just a diagram-free way put $n-1$ evenly spaced points around a circle.

Now we know how to schedule a round-robin tournament between n people in just $n-1$ rounds, if n is even!

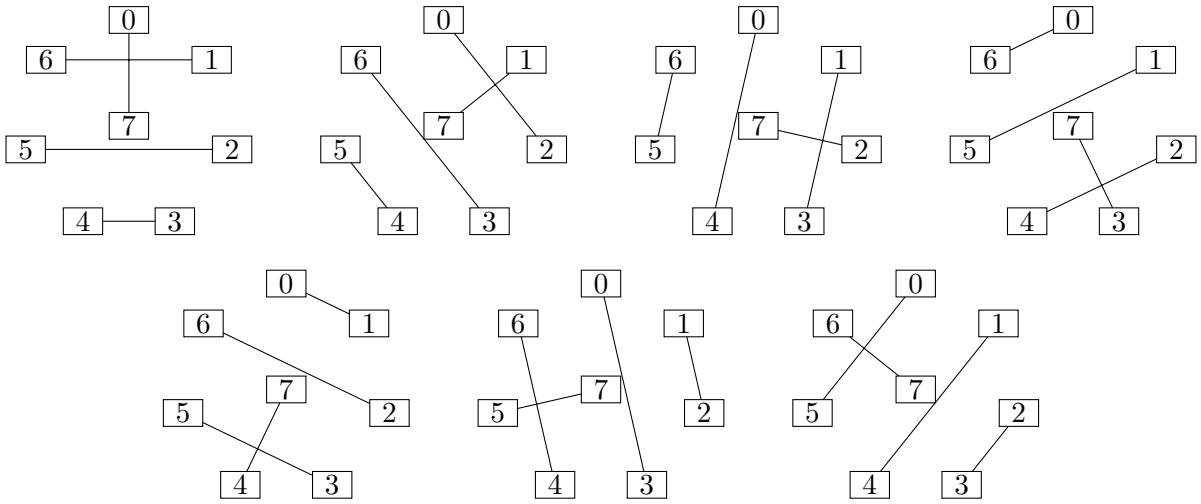


Figure 15.5: The $n = 8$ case of Theorem 15.3

Question:	What do tournament organizers do if n is odd?
Answer:	In each round, one player gets a "bye" and sits out. This is equivalent to scheduling an $(n+1)$-player tournament with one player named "bye" who doesn't really exist.

Since $n+1$ is even whenever n is odd, adding a fictional player named "bye" reduces the problem to a case where Theorem 15.3 applies. Of course, with $n+1$ players, we require n rounds, even if one of the players is fictional. This proves the following corollary:

Corollary 15.4. *When n is odd, the complete graph K_n has a decomposition into n matchings, which are each nearly perfect (covering $n-1$ of the n vertices).*

15.5 Increasing walks

The problem of finding a 1-factorization of K_n was historically first studied by tournament organizers, not graph theorists. However, that does not mean it is not useful in graph theory. I encountered the following problem as a graduate student.

Take the complete graph K_n, and make it into a weighted graph by giving each edge e a weight $w(e)$: in this problem, we will insist that all the weights should be different. A walk $x_0, x_1, x_2, \ldots, x_k$ is called an **increasing walk** if the weights of the edges $x_0x_1, x_1x_2, \ldots, x_{k-1}x_k$ are sorted in ascending order: in other words, if the weights go up as we walk.

What is the longest possible increasing walk? Well, it depends on the labels. We can have very long increasing walks if the weights cooperate: pick a walk in which no edges repeat, and give the i^{th} edge along this walk weight i.

Question:	What is the longest walk in K_n that does not repeat any edges?
Answer:	If n is odd, then all degrees are even, so K_n has an Euler tour: a walk of length $\binom{n}{2}$. If n is even, then we can delete any matching and be left with an Eulerian graph, which has a walk of length $\binom{n}{2} - n/2$.

It is more interesting to ask the question: what is the longest possible length of an increasing walk that we can guarantee, no matter what the weights of the edges are?

This problem was first studied by Ron Graham and Daniel Kleitman, who proved in 1973 [1] that it was possible to find weighted graphs in which the longest increasing walk is much shorter.

Proposition 15.5. *For all even n, there is a weighted complete graph on n vertices in which no increasing walk has length more than $n-1$.*

Proof. Use Theorem 15.3 to decompose K_n into $n-1$ perfect matchings, and give weight i to every edge in the i^{th} perfect matching M_i. Okay, that doesn't quite work, because the weights all have to be different, but it will have the same effect if we give every edge in M_i some weight in the interval $[i, i+\frac{1}{2}]$: we will never end up comparing two edges in a matching anyway.

An increasing walk in this weighted graph cannot use more than one edge from any matching M_i. After taking its first edge from that matching, it cannot immediately take a second edge, because no two edges in M_i share an endpoint. So the walk must follow up by going to a different matching: since the walk is increasing, it must select an edge from M_j, for some $j > i$. But this edge has a greater weight than any edge of M_i, so the increasing walk can never return to M_i again.

Since there are only $n-1$ matchings, the increasing walk cannot use more than $n-1$ edges. □

Graham and Kleitman proved more than this. They generalized Proposition 15.5 to work for odd n as well, excluding the special case $n = 3$ and $n = 5$. Moreover, they proved that this upper bound is always achievable! I will present their solution in a different style, as described by Peter Winkler [12] and attributed to Ehud Friedman.

Proposition 15.6. *In every weighted complete graph on n vertices, there is an increasing walk of length at least $n-1$.*

Proof. Imagine that we put n different people on the n vertices of the complete graph. Then, we call out the edges of the graph one by one, in increasing order of weight. Whenever we call out an edge xy, the two people standing on vertices x and y trade places, walking along edge xy in opposite directions.

At the end, we will ask each person to describe the walk they took. All these walks must be increasing, because all edges were announced in increasing order. Let $\ell_1, \ell_2, \ldots, \ell_n$ be the lengths of the n walks.

Question:	How can we express the sum $\ell_1 + \ell_2 + \cdots + \ell_n$ in another way?
Answer:	The sum counts each edge of the graph twice, since two people walk each edge, so it is equal to $2\binom{n}{2}$ or $n(n-1)$.

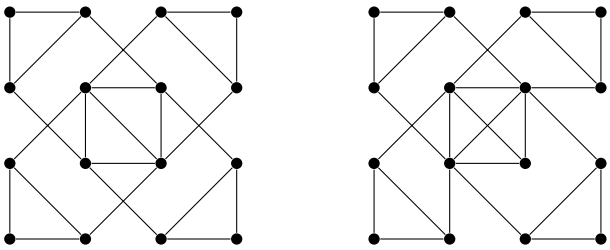
If the lengths $\ell_1, \ell_2, \ldots, \ell_n$ add up to $n(n-1)$, then their average is

$$\frac{\ell_1 + \ell_2 + \cdots + \ell_n}{n} = \frac{n(n-1)}{n} = n - 1,$$

and it is impossible for all n lengths to be below average. Therefore at least one walk must have length at least $n - 1$. □

15.6 Practice problems

1. In one of the graphs below, find a perfect matching. In the other, prove that there is no perfect matching, by finding a Tutte set.



2. Theorem 14.2 from the previous chapter implies that every 3-regular bipartite graph has a perfect matching.

   Prove that the word "bipartite" cannot be left out: give an example of a 3-regular graph which is *not* bipartite, and does *not* have a perfect matching.

3. Use Theorem 15.1 (Tutte's theorem) to prove Theorem 14.1 (Hall's theorem).

4. Let $o(G)$ denote the number of odd components in a graph G. Prove that if G is a graph and U is a subset of $V(G)$, then

$$\alpha'(G) \leq \frac{1}{2}\big(|V(G)| - o(G - U) + |U|\big).$$

   (More is true. The Tutte–Berge formula is a generalization of Theorem 15.1 saying that the two sides of this inequality are equal for some set U.)

5. Suppose that $2n$ people from two different n-person teams participate in a tournament. Prove that it is possible to schedule n rounds of simultaneous games between players on opposing teams, so that each person plays once against everyone on the other team.

   (In other words, find a 1-factorization of the complete bipartite graph $K_{n,n}$.)

6. Let G be a copy of K_8 with vertices $\{000, 001, \ldots, 111\}$ (just like the vertices of the cube graph Q_3).

a) For each nonempty subset $S \subseteq \{1, 2, 3\}$, let M_S be the set of all edges in G whose endpoints differ in the positions numbered by S. For example,

$$M_{\{1,2\}} = \Big\{\{000, 110\}, \{001, 111\}, \{010, 100\}, \{011, 111\}\Big\}.$$

Prove that the seven matchings $M_{\{1\}}, M_{\{2\}}, M_{\{3\}}, M_{\{1,2\}}, M_{\{1,3\}}, M_{\{2,3\}}, M_{\{1,2,3\}}$ are a 1-factorization of G.

b) Prove that this 1-factorization is not isomorphic to the one found in Theorem 15.3. That is, prove that it is not possible to replace the labels $\{000, 001, \ldots, 111\}$ by $\{0, 1, \ldots, 7\}$ in any way to turn this 1-factorization into the one shown in Figure 15.5.

7. An **increasing path** is simply an increasing walk which is a path: it does not repeat any vertices.

We can prove a version of Proposition 15.6 for increasing paths by changing the rules slightly in that proof. Whenever edge xy is called out, if the person on x has already visited y, or the person on y has already visited x, then both people stay put. This ensures that at the end, each person's walk is an increasing path.

a) Suppose that each person walks a path of length at most L. Prove that each person is responsible for "rejecting" at most $\frac{L(L-1)}{2}$ edges.

b) At the end, each of the $\binom{n}{2}$ edges was either walked by two people or rejected by at least one person. Use this to prove the inequality $n \cdot \frac{L}{2} + n \cdot \frac{L(L-1)}{2} \geq \binom{n}{2}$.

c) Prove $L \geq \sqrt{n-1}$.

Graham and Kleitman proved a similar result, but unlike the problem of increasing walks, the exact worst-case answer is still not known in the case of increasing paths.

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